



**HAL**  
open science

# Fractional Supersymmetry as a Superposition of Ordinary Supersymmetry

M. Daoud, M. Kibler

► **To cite this version:**

M. Daoud, M. Kibler. Fractional Supersymmetry as a Superposition of Ordinary Supersymmetry. X International Conference on Symmetry Methods in Physics, Aug 2003, Yerevan, Armenia. pp.1654-1656. in2p3-00024531

**HAL Id: in2p3-00024531**

**<https://hal.in2p3.fr/in2p3-00024531>**

Submitted on 5 Sep 2005

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

M. Daoud, M. Kibler

## Fractional Supersymmetry as a Superposition of Ordinary Supersymmetry<sup>1</sup>

A B S T R A C T

It is shown how to derive fractional supersymmetric quantum mechanics of order  $k$  as a superposition of  $k - 1$  copies of ordinary supersymmetric quantum mechanics.

---

<sup>1</sup>Paper written from a communication by M. Kibler at the “X International Conference on Symmetry Methods in Physics” (Yerevan, Armenia, 13 - 19 August 2003). To be published in Phys. Atom. Nuclei (Yadernaya fizika).

# FRACTIONAL SUPERSYMMETRY AS A SUPERPOSITION OF ORDINARY SUPERSYMMETRY

M. Daoud<sup>†</sup> M. Kibler<sup>‡</sup>

September 5, 2005

## 1 Introduction

In recent years, fractional supersymmetry has been the subject of numerous works. Indeed,  $k$ -fractional supersymmetry is closely connected to the notion of quantum algebra (deformation theory) and to the concept of intermediate statistics (of anyons [1] and  $k$ -fermions [2, 3]) interpolating between Bose-Einstein statistics and Fermi-Dirac statistics. Therefore, fractional supersymmetry constitutes a useful tool for dealing with anyonic statistics.

Fractional supersymmetric quantum mechanics of order  $k$  can be considered as an extension of ordinary supersymmetric quantum mechanics which corresponds to  $k = 2$ . An ordinary supersymmetric quantum-mechanical system may be generated from a doublet  $(H, Q)_2$  of operators satisfying [4, 5]

$$Q^2 = 0,$$

$$QQ^\dagger + Q^\dagger Q = H.$$

The self-adjoint operator  $H$  and the operator  $Q$  act on a separable Hilbert space. The operator  $H$  is referred to as the Hamiltonian and the operator  $Q$  as the supersymmetry operator of the ordinary supersymmetric quantum-mechanical system. The operator  $Q$  gives rise to  $\mathcal{N} = 2$  dependent supercharges  $Q_- = Q$  and  $Q_+ = Q^\dagger$  connected via Hermitean conjugation. They are nilpotent operators of order  $k = 2$  and commute with the Hamiltonian  $H$ .

The *ordinary* supersymmetric quantum-mechanical system  $(H, Q)_2$  can be extended to a *fractional* supersymmetric quantum-mechanical system  $(H, Q)_k$  with  $k \in \mathbf{N} \setminus \{0, 1, 2\}$

---

<sup>†</sup>Laboratoire de Physique de la Matière Condensée, Faculté des Sciences, Université Ibn Zohr, BP 28/S, Agadir, Morocco

<sup>‡</sup>Institut de Physique Nucléaire de Lyon, IN2P3-CNRS et Université Claude Bernard, 43 Bd du 11 Novembre 1918, F-69622 Villeurbanne Cedex

as follows. The system  $(H, Q)_k$  may be defined by [6, 7]

$$Q_- = Q, \quad Q_+ = Q^\dagger \quad (\Rightarrow Q_+ = Q_-^\dagger), \quad Q_\pm^k = 0, \quad (1a)$$

$$Q_-^{k-1}Q_+ + Q_-^{k-2}Q_+Q_- + \cdots + Q_+Q_-^{k-1} = Q_-^{k-2}H, \quad (1b)$$

$$[H, Q_\pm] = 0, \quad H = H^\dagger, \quad (1c)$$

where the self-adjoint operator  $H$ , the Hamiltonian of the system, and the  $\mathcal{N} = 2$  supercharges  $Q_-$  and  $Q_+$  act on a separable Hilbert space. Of course, the case  $k = 2$  corresponds to an ordinary supersymmetric quantum-mechanical system.

In the present work, we study how it is possible to connect ordinary and  $k$ -fractional supersymmetric quantum-mechanical systems.

## 2 The algebra $W_k$

As an interesting question, we may ask: How to construct a fractional supersymmetric quantum-mechanical system of order  $k$  and, thus, fractional supersymmetric quantum mechanics of order  $k$ ? This question can be answered through the definition of a generalized Weyl-Heisenberg algebra  $W_k$ . We now define the generic algebra  $W_k$  and shall see in the next section how a fractional supersymmetric quantum-mechanical system of order  $k$  may be associated with a given algebra  $W_k$ .

For  $k$  given, with  $k \in \mathbf{N} \setminus \{0, 1\}$ , the algebra  $W_k$  is generated by four linear operators  $X_-$ ,  $X_+$ ,  $N$  and  $K$ . The operators  $X_-$  and  $X_+ = X_-^\dagger$  are shift operators connected via Hermitean conjugation. The operator  $N$ , called number operator, is self-adjoint. Finally, the operator  $K$  is a  $Z_k$ -grading unitary operator. The generators  $X_-$ ,  $X_+$ ,  $N$  and  $K$  satisfy [8]

$$\begin{aligned} [X_-, X_+] &= \sum_{s=0}^{k-1} f_s(N) \Pi_s, \\ [N, X_-] &= -X_-, \quad (+\text{h.c.}), \\ [K, X_-]_q &= 0, \quad (+\text{h.c.}), \\ [K, N] &= 0, \quad K^k = 1. \end{aligned}$$

The functions  $f_s : N \mapsto f_s(N)$  are such that  $f_s(N)^\dagger = f_s(N)$ ,  $[A, B]_q$  stands for  $AB - qBA$ , and the operators  $\Pi_s$  are defined by

$$\Pi_s = \frac{1}{k} \sum_{t=0}^{k-1} q^{-st} K^t$$

where

$$q = \exp\left(\frac{2\pi i}{k}\right)$$

is a root of unity. To a given set  $\{f_s : s = 0, 1, \dots, k-1\}$  corresponds one algebra  $W_k$ .

The generalized Weyl-Heisenberg algebra  $W_k$  covers numerous algebras describing exactly solvable one-dimensional systems. The particular system corresponding to a given

set  $\{f_s : s = 0, 1, \dots, k-1\}$  yields, in a Schrödinger picture, a particular dynamical system with a specific potential. Let us mention two interesting cases. The case

$$\forall s \in \{0, 1, \dots, k-1\} : f_s(N) = f_s \text{ independent of } N$$

corresponds to systems with cyclic shape-invariant potentials (in the sense of Ref. [9]) and the case

$$\forall s \in \{0, 1, \dots, k-1\} : f_s(N) = aN + b, \quad (a, b) \in \mathbf{R}^2$$

to systems with translational shape-invariant potentials (in the sense of Ref. [10]). For instance, the case  $(a = 0, b > 0)$  corresponds to the harmonic oscillator potential, the case  $(a < 0, b > 0)$  to the Morse potential and the case  $(a > 0, b > 0)$  to the Pöschl-Teller potential. For these various potentials, the part of  $W_k$  spanned by  $X_-$ ,  $X_+$  and  $N$  can be identified with the ordinary Weyl-Heisenberg algebra for  $(a = 0, b \neq 0)$ , with the  $\text{su}(2)$  Lie algebra for  $(a < 0, b > 0)$  and with the  $\text{su}(1,1)$  Lie algebra for  $(a > 0, b > 0)$ .

### 3 A $k$ -fractional system associated with $W_k$

In order to associate a  $k$ -fractional supersymmetric quantum-mechanical system associated with a given generalized Weyl-Heisenberg algebra  $W_k$ , we must define a supersymmetry operator  $Q$  and an Hamiltonian  $H$ . The supersymmetry operator  $Q$  is defined by

$$Q \equiv Q_- = X_-(1 - \Pi_1) \Leftrightarrow Q^\dagger \equiv Q_+ = X_+(1 - \Pi_0).$$

Then, the Hamiltonian  $H$  associated with  $W_k$  can be deduced from Eq. (1b). This yields

$$H = (k-1)X_+X_- - \sum_{s=3}^k \sum_{t=2}^{s-1} (t-1) f_t(N-s+t) \Pi_s \\ - \sum_{s=1}^{k-1} \sum_{t=s}^{k-1} (t-k) f_t(N-s+t) \Pi_s.$$

(Note that the summation from  $s = k-2$  to  $s = k$  appearing in some previous works by the authors [8] should be replaced by a summation from  $s = 3$  to  $s = k$ .) It can be checked that  $H$  is self-adjoint and commutes with  $Q_-$  and  $Q_+$ . As a conclusion, the doublet  $(H, Q)_k$  associated to  $W_k$  satisfies Eq. (1) and thus defines a  $k$ -fractional supersymmetric quantum-mechanical system.

### 4 Connection between fractional supersymmetry and ordinary supersymmetry

In order to establish a connection between *fractional* supersymmetric quantum mechanics of order  $k$  and *ordinary* supersymmetric quantum mechanics (corresponding to  $k = 2$ ), it is necessary to construct sub-systems from the doublet  $(H, Q)_k$  that correspond to ordinary

supersymmetric quantum-mechanical systems. This may be achieved in the following way [11]. The general Hamiltonian  $H$  can be rewritten as

$$H = \sum_{s=1}^k H_s \Pi_s$$

where

$$H_s \equiv H_s(N) = (k-1)X_+X_- - \sum_{t=2}^{k-1} (t-1) f_t(N-s+t) \\ + (k-1) \sum_{t=s}^{k-1} f_t(N-s+t).$$

It can be shown that the operators  $H_k \equiv H_0, H_{k-1}, \dots, H_1$  turn out to be isospectral operators. It is possible to factorize  $H_s$  as [11]

$$H_s = X(s)_+ X(s)_-$$

Let us now define: (i) the two (supercharge) operators

$$q(s)_- = X(s)_- \Pi_s, \quad q(s)_+ = X(s)_+ \Pi_{s-1}$$

and (ii) the (Hamiltonian) operator

$$h(s) = X(s)_- X(s)_+ \Pi_{s-1} + X(s)_+ X(s)_- \Pi_s.$$

It is then a simple matter of calculation to prove that  $h(s)$  is self-adjoint and that

$$q(s)_+ = q(s)_-^\dagger, \quad q(s)_\pm^2 = 0, \quad h(s) = \{q(s)_-, q(s)_+\}, \quad [h(s), q(s)_\pm] = 0.$$

Consequently, the doublet  $(h(s), q(s))_2$ , with  $q(s) \equiv q(s)_-$ , satisfies Eq. (1) with  $k = 2$  and thus defines an ordinary supersymmetric quantum-mechanical system (corresponding to  $k = 2$ ).

The Hamiltonian  $h(s)$  is amenable to a form more appropriate for discussing the link between ordinary supersymmetry and fractional supersymmetry. Indeed, we can show that

$$X(s)_- X(s)_+ = H_s(N+1).$$

Then, we can obtain the important relation

$$h(s) = H_{s-1} \Pi_{s-1} + H_s \Pi_s$$

to be compared with the expansion of  $H$  in terms of supersymmetric partners  $H_s$ .

As a result, the system  $(H, Q)_k$ , corresponding to  $k$ -fractional supersymmetry, can be described in terms of  $k-1$  sub-systems  $(h(s), q(s))_2$ , corresponding to ordinary supersymmetry. The Hamiltonian  $h(s)$  is given as a sum involving the supersymmetric partners  $H_{s-1}$  and  $H_s$ . Since the supercharges  $q(s)_\pm$  commute with the Hamiltonian  $h(s)$ , it follows that

$$H_{s-1}X(s)_- = X(s)_-H_s, \quad H_sX(s)_+ = X(s)_+H_{s-1}.$$

As a consequence, the operators  $X(s)_+$  and  $X(s)_-$  render possible to pass from the spectrum of  $H_{s-1}$  and  $H_s$  to the one of  $H_s$  and  $H_{s-1}$ , respectively. This result is quite familiar for ordinary supersymmetric quantum mechanics (corresponding to  $s = 2$ ).

For  $k = 2$ , the operator  $h(1)$  is nothing but the total Hamiltonian  $H$  corresponding to ordinary supersymmetric quantum mechanics. For arbitrary  $k$ , the other operators  $h(s)$  are simple replicas (except for the ground state of  $h(s)$ ) of  $h(1)$ . In this sense, fractional supersymmetric quantum mechanics of order  $k$  can be considered as a set of  $k - 1$  replicas of ordinary supersymmetric quantum mechanics corresponding to  $k = 2$  and typically described by  $(h(s), q(s))_2$ . As a further argument, it is to be emphasized that

$$H = q(2)_- q(2)_+ + \sum_{s=2}^k q(s)_+ q(s)_-$$

which can be identified with  $h(2)$  for  $k = 2$ .

## 5 Conclusions

Starting from a  $Z_k$ -graded algebra  $W_k$ , characterized by a set  $\{f_s : s = 0, 1, \dots, k - 1\}$  of structure functions, it was shown how to associate a  $k$ -fractional supersymmetric quantum-mechanical system  $(H, Q)_k$  characterized by an Hamiltonian  $H$  and a supercharge  $Q$ .

The Hamiltonian  $H$  for the system  $(H, Q)_k$  was developed as a superposition of  $k$  isospectral supersymmetric partners  $H_k, H_{k-1}, \dots, H_1$ . It was proved that the system  $(H, Q)_k$  can be described in terms of  $k - 1$  sub-systems  $(h(s), q(s))_2$  which are ordinary supersymmetric quantum-mechanical systems.

## References

- [1] Goldin, G.A., Menikoff, R., and Sharp, D.H., *J. Math. Phys.*, 1980, vol.21, p.650.  
Goldin, G.A., and Sharp, D.H., *Phys. Rev. Lett.*, 1996, vol.76, p.1183.
- [2] Daoud, M., Hassouni, Y., and Kibler, M., in: *Symmetries in Science X*, eds. Gruber, B., and Ramek, M., Plenum, New York, 1998; *Yad. Fiz.*, 1998, vol.61, p.1935.
- [3] Pan, H.-Y., and Zhao, Z.S., *Phys. Lett. A*, 2003, vol.312, p.1.
- [4] Nicolai, H., *J. Phys. A: Math. Gen.*, 1976, vol.9, p.1497.
- [5] Witten, E., *Nucl. Phys. B*, 1981, vol.188, p.513; 1982, vol.202, p.253.
- [6] Rubakov, V.A., and Spiridonov, V.P., *Mod. Phys. Lett. A*, 1988, vol.3, p.1337.
- [7] Khare, A., *J. Phys. A*, 1992, vol.25, p.L749; *J. Math. Phys.*, 1993, vol.34, p.1277.
- [8] Daoud, M., and Kibler, M., *Phys. Part. Nuclei (Suppl. 1)*, 2002, vol.33, p.S43; *Int. J. Quantum Chem.*, 2003, vol.91, p.551.
- [9] Sukhatme, U.P., Rasinariu, C., and Khare, A., *Phys. Lett. A*, 1997, vol.234, p.401.
- [10] Junker, G., *Supersymmetric Methods in Quantum and Statistical Physics*, Springer, Berlin, 1996.
- [11] Daoud, M., and Kibler, M., *Phys. Lett. A*, 2004, vol.321, p.147.