Symmetries in Nuclei

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Abstract

The use of dynamical symmetries or spectrum generating algebras for the solution of the nuclear many-body problem is reviewed. General notions of symmetry and dynamical symmetry in quantum mechanics are introduced and illustrated with simple examples such as the SO(4) symmetry of the hydrogen atom and the isospin symmetry in nuclei. Two nuclear models, the shell model and the interacting boson model, are reviewed with particular emphasis on their use of group-theoretical techniques.

PACS numbers: 03.65.Fd, 21.60.Fw, 21.60.Cs, 21.60.Ev

I. INTRODUCTION

In the *Oxford Dictionary of Current English* symmetry is defined as the 'right correspondence of parts; quality of harmony or balance (in size, design etc.) between parts'. The word is derived from Greek where it has the meaning 'with proportion' or 'with order'. In modern theories of physics it has acquired a more precise meaning but the general idea of seeking to order physical phenomena still remains. Confronted with the bewildering complexity exhibited by the multitude of physical systems, physicists attempt to extract some simple regularities from observations, and the fact that they can do so is largely due to the presence of symmetries in the laws of physics. Although one can never hope to explain all observational complexities entirely on the basis of symmetry arguments alone, these are nevertheless instrumental in establishing correlations between and (hidden) regularities in the data.

The mathematical theory of symmetry is called group theory and its origin dates back to the nineteenth century. Of course, the notion of symmetry is present implicitly in many mathematical studies that predate the birth of group theory and goes back even to the ancient Greeks, in particular Euclid. It was, however, Évariste Galois who perceived the importance of the group of permutations to answer the question whether the roots of a polynomial equation can be algebraically represented or not. (A readable summary of the solution of this problem is given in the first chapter of Gilmore's book [1].) In the process of solving that long-standing mathematical problem he invented group theory as well as Galois theory which studies the relation between polynomials and groups. The mathematical theory of groups developed further throughout the nineteenth century and made another leap forward in 1873 when Sophus Lie proposed the concept of a Lie group and its associated Lie algebra.

For a long time it was assumed that group theory was a branch of mathematics without any application in the physical sciences. This state of affairs changed with the advent of quantum mechanics, and it became clear that group theory provides a powerful tool to understand the structure of quantum systems from a unified perspective. After the introduction of symmetry transformations in abstract spaces (associated, for example, with isospin, flavor, color, etc.) the role of group theory became even central.

The purpose of these lecture notes is to introduce, explain and illustrate the concepts of symmetry and dynamical symmetry. In Sect. II a brief reminder is given of the central role of symmetry in quantum mechanics and of its relation with invariance and degeneracy. There exist two standard examples to illustrate the idea that symmetry implies degeneracy and *vice versa*, namely the hydrogen atom and the harmonic oscillator. In Sect. III the first of them is analyzed in detail. Section IV describes the process of symmetry breaking and, in particular, dynamical symmetry breaking in the sense as it is used in these lecture notes. This mechanism is illustrated in Sect. V with a detailed example, namely isospin and its breaking in nuclei. Sections VI and VII then present the nuclear shell model and the interacting boson model, respectively, with a special emphasis on the symmetry techniques that have been used in the context of these models. Finally, in Sect. VIII, a summary of these lecture notes is given.

II. SYMMETRY IN QUANTUM MECHANICS

The starting point of any discussion of symmetry is that the laws of physics should be invariant with respect to certain transformations of the reference frame, such as a translation or rotation, or a different choice of the origin of the time coordinate. This observation leads to three fundamental conservation laws: conservation of linear momentum, angular momentum and energy. In some cases an additional space-inversion symmetry applies, yielding another conserved quantity, namely parity. In a relativistic framework the above transformations on space and time cannot be considered separately but become intertwined. The laws of nature are then invariant under the Lorentz transformations which operate in four-dimensional space–time.

These transformations and their associated invariances can be called 'geometric' in the sense that they are defined in space–time. In quantum mechanics, an important extension of these concepts is obtained by also considering transformations that act in abstract spaces associated with intrinsic variables such as spin, isospin (in atomic nuclei), flavor and color (of quarks) etc. It is precisely these 'intrinsic' invariances which have lead to the preponderance of symmetry applications in the quantum physics.

To be more explicit, consider a transformation acting on a physical system, that is, an operation that transforms the coordinates $\bar{r}_i$ and the momenta $\bar{p}_i$ of the particles that constitute the system. Such transformations are of a geometric nature. For a discussion of symmetry in quantum-mechanical systems this definition is too restrictive and the appropriate generalization is to consider, instead of the geometric transformations themselves, the corresponding transformations in the Hilbert space of quantum-mechanical states of the system. The action of the geometric transformation on spin variables (*i.e.*, components of the spin vector) is assumed to be identical to its action on the components of the angular momentum vector $\bar{\ell} = \bar{r} \wedge \bar{p}$. Furthermore, it can be shown [2] that a correspondence exists between the geometric transformations in physical space and the transformations induced by it in the Hilbert space of quantum-mechanical states. This correspondence, however, is not necessarily one-to-one; that is only the case if the system is 'bosonic' (consists of any number of integer-spin bosons and/or an even number of half-integer-spin fermions). If the system is 'fermionic' (contains an odd number of fermions), the correspondence is two-to-one and the groups, formed by the geometric transformations and by the corresponding transformations in the Hilbert space of quantum-mechanical states, are not isomorphic but rather homomorphic.

No distinction is made in the following between geometric and quantum-mechanical transformations; all elements g_i will be taken as operators acting on the Hilbert space of quantum-mechanical states.

A. Symmetry

A time-independent Hamiltonian H which commutes with the generators g_k that form a Lie algebra G,

$$\forall g_k \in \mathrm{G} : [H, g_k] = 0, \tag{1}$$

is said to have a symmetry G or, alternatively, to be invariant under G. The determination of operators g_k that leave invariant the Hamiltonian of a given physical system is central to any quantum-mechanical description. The reasons for this are profound and can be understood from the correspondence between geometrical and quantum-mechanical transformations. It can be shown [2] that the transformations g_k with the symmetry property (1) are induced by geometrical transformations that leave unchanged the corresponding classical Hamiltonian. In this way the classical notion of a conserved quantity is transcribed in quantum mechanics in the form of the symmetry property (1) of the time-independent Hamiltonian.

B. Degeneracy and state labeling

A well-known consequence of a symmetry is the occurrence of degeneracies in the eigen-spectrum of H. Given an eigenstate $|\gamma\rangle$ of H with energy E, the condition (1) implies that the states $g_k|\gamma\rangle$ all have the same energy,

$$Hg_k|\gamma\rangle = g_kH|\gamma\rangle = Eg_k|\gamma\rangle. \tag{2}$$

An arbitrary eigenstate of H shall be written as $|\Gamma\gamma\rangle$, where the first quantum number Γ is different for states with different energies and the second quantum number γ is needed to label degenerate eigenstates. The eigenvalues of a Hamiltonian that satisfies (1) depend on Γ only,

$$H|\Gamma\gamma\rangle = E(\Gamma)|\Gamma\gamma\rangle, \tag{3}$$

and, furthermore, the transformations g_k do not admix states with different Γ,

$$g_k|\Gamma\gamma\rangle = \sum_{\gamma'} a^{\Gamma}_{\gamma'\gamma}(k)|\Gamma\gamma'\rangle. \tag{4}$$

This simple discussion of the consequences of a Hamiltonian symmetry illustrates the relevance of group theory in quantum mechanics. Symmetry implies degeneracy and eigenstates that are degenerate in energy provide a Hilbert space in which irreducible representations of the symmetry group are constructed. Consequently, the irreducible representations of a given group directly determine the degeneracy structure of a Hamiltonian with the symmetry associated to that group.

Eigenstates of H can be denoted as $|\Gamma\gamma\rangle$ where the symbol Γ labels the irreducible representations of G. Note that the same irreducible representation might occur more than once in the eigenspectrum of H and, therefore, an additional multiplicity label η should be introduced to define a complete labeling of eigenstates as $|\eta\Gamma\gamma\rangle$. This label shall be omitted in the subsequent discussion.

A sufficient condition for a Hamiltonian to have the symmetry property (1) is that it is a Casimir operator which by definition commutes with all generators of the algebra. The eigenequation (3) then becomes

$$C_m[\mathrm{G}]|\Gamma\gamma\rangle = E_m(\Gamma)|\Gamma\gamma\rangle. \tag{5}$$

In fact, all results remain valid if the Hamiltonian is an analytic function of Casimir operators of various orders. The energy eigenvalues $E_m(\Gamma)$ are functions of the labels that specify the irreducible representation Γ, and are known for all classical Lie algebras [3].

These concepts can be illustrated with the example of the hydrogen atom which is discussed in detail in the next section.

III. THE HYDROGEN ATOM

The Hamiltonian for a particle of charge $-e$ and mass m_e in a Coulomb potential e/r is given by

$$H_\mathrm{H} = \frac{p^2}{2m_\mathrm{e}} - \frac{e^2}{r} = -\frac{\hbar^2}{2m_\mathrm{e}}\nabla^2 - \frac{e^2}{r}. \tag{6}$$

This is taken here as a model Hamiltonian for the hydrogen atom. The Hamiltonian is independent of the spin of the electron which leads to a two-fold degeneracy of all states corresponding to spin-up and spin-down. Electron spin is ignored in the following and the symmetry properties of the spatial part only of the electron wave function are studied.

The solutions of the associated Schrödinger equation, $H_\mathrm{H}\tilde{\phi}(\bar{r}) = E\tilde{\phi}(\bar{r})$, are well known from standard quantum mechanics. The energies of the stationary states are

$$E(n) = -\frac{m_\mathrm{e}e^4}{2\hbar^2 n^2} \equiv -\frac{R_\mathrm{H}}{n^2}, \tag{7}$$

where R_H is the Rydberg constant and n the so-called principal quantum number. The electron wave functions are

$$\tilde{\phi}_{n\ell m_\ell}(r,\theta,\varphi) = \tilde{R}_{n\ell}(r)Y_{\ell m_\ell}(\theta,\varphi), \tag{8}$$

with $\tilde{R}_{n\ell}(r)$ and $Y_{\ell m_\ell}(\theta,\varphi)$ known functions[49]. The $Y_{\ell m_\ell}(\theta,\varphi)$ are spherical harmonics which occur for any central potential with spherical symmetry. The $\tilde{R}_{n\ell}(r)$ are radial wave functions whose exact form is not of concern here. The solution of the differential equation $H_\mathrm{H}\tilde{\phi}(\bar{r}) = E\tilde{\phi}(\bar{r})$ also leads to the conditions

$$n = 1, 2, \ldots, \qquad \ell = 0, 1, \ldots, n-1, \qquad m_\ell = -\ell, -\ell+1, \ldots, +\ell. \tag{9}$$

The energy spectrum of the hydrogen atom is shown in Fig. 1. The energy eigenvalues $E(n)$ only depend on n and not on ℓ or m_ℓ. A given level with energy $E(n)$ is thus n^2-fold

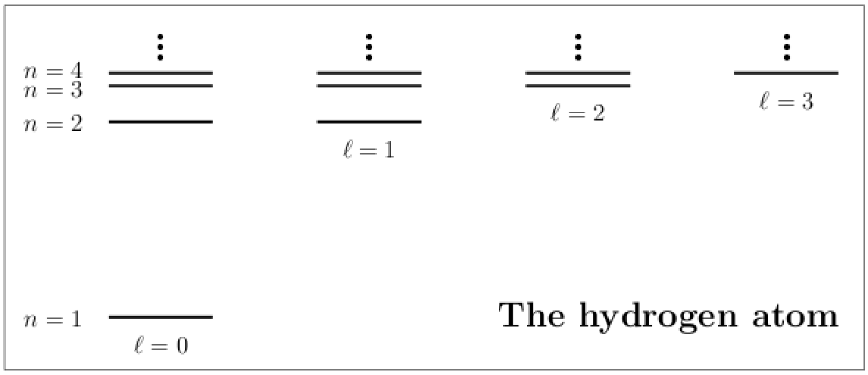

FIG. 1: The energy spectrum of the hydrogen atom.

degenerate since

$$\sum_{\ell=0}^{n-1}(2\ell+1)=n^2. \tag{10}$$

The nature of this degeneracy will be explained using symmetry arguments and, in addition, it will be shown that the entire spectrum can be determined with algebraic methods without recourse to boundary conditions of differential equations.

The Hamiltonian of the hydrogen atom is rotationally [or SO(3)] invariant. This is obvious on intuitive grounds since the properties of the hydrogen atom do not change under rotation. Formally, it follows from the following commutation property:

$$[H_{\mathrm{H}}, L_\mu]=0, \tag{11}$$

where L_μ are the components of the angular momentum operator[50], $\bar{L}=(\bar{r}\wedge\bar{p})=-i\hbar(\bar{r}\wedge\bar{\nabla})$. It is of interest to look more closely at the origin of the vanishing commutator between H_{H} and L_μ. The Hamiltonian of the hydrogen atom consists of two parts, kinetic and potential, and *both* commute with L_μ since

$$[\nabla^2, L_\mu]=0, \qquad [r^{-1}, L_\mu]=0, \tag{12}$$

where use is made of commutation relations like

$$[\bar{\nabla}, r^k]=k\, r^{k-2}\bar{r}, \quad [\nabla^2, \bar{r}]=2\bar{\nabla}. \tag{13}$$

Since the components L_μ form an SO(3) algebra,

$$[L_\mu, L_\nu]=i\hbar\sum_{\rho=1}^{3}\epsilon_{\mu\nu\rho}L_\rho, \tag{14}$$

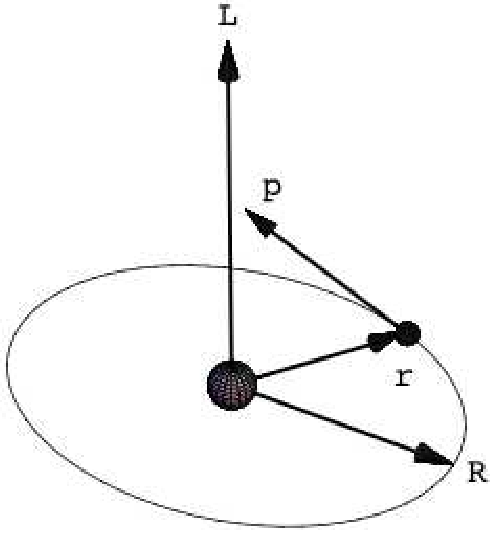

FIG. 2: The angular momentum vector $\bar{L}$ and the Runge–Lenz vector $\bar{R}$ in the classical Kepler problem of a planet orbiting the sun.

and since L_μ commutes with $H_{\rm H}$, one concludes that the Hamiltonian of the hydrogen atom has an SO(3) symmetry. This explains part of the observed degeneracy, namely, levels with a given ℓ are $(2\ell+1)$-fold degenerate.

To understand the origin of the *complete* degeneracy of the hydrogen spectrum, it is instructive to consider first the Kepler problem of the motion of a single planet around the sun which is the classical analogue of the hydrogen atom. Besides angular momentum, there is another conserved quantity because there is no precession of the planetary orbit, that is, the major axis of its elliptic trajectory is fixed. In contrast to the conservation of angular momentum which is valid for *all* central potentials, the absence of precession is a specific property of the Newtonian $1/r$ potential. The associated conserved quantity is known from classical mechanics,

$$\bar{R}_{\rm cl} = \frac{\bar{p}\wedge\bar{L}}{m_{\rm e}} - e^2\frac{\bar{r}}{r}. \tag{15}$$

This vector is known as the Runge–Lenz (or also Lenz–Pauli) vector and its three components are conserved for a $1/r$ potential, that is, not only its direction (along the major axis of the orbit) but also its magnitude is conserved (see Fig. 2). The latter property follows from the relation

$$\bar{R}_{\rm cl}^2 = e^4 + \frac{2E}{m_{\rm e}}\bar{L}^2, \tag{16}$$

which shows that $\bar{R}_{\rm cl}^2$ can be expressed in terms of the energy and the angular momentum, both of which are conserved.

The construction of the quantum-mechanical equivalent of the Runge–Lenz vector is done in the usual way and yields

$$\bar{R}' = -\frac{\hbar^2}{2m_{\mathrm{e}}}[\bar{\nabla} \wedge (\bar{r} \wedge \bar{\nabla}) - (\bar{r} \wedge \bar{\nabla}) \wedge \bar{\nabla}] - e^2 \frac{\bar{r}}{r}. \tag{17}$$

The relation (16) between the energy and the moduli of the angular momentum and Runge–Lenz vectors converts to

$$\bar{R}'^2 = e^4 + \frac{2H_{\mathrm{H}}}{m_{\mathrm{e}}} \left(\bar{L}^2 + \hbar^2 \right). \tag{18}$$

From the classical analysis one expects R'_μ to commute with H_{H},

$$[H_{\mathrm{H}}, R'_\mu] = 0, \tag{19}$$

which is indeed confirmed through explicit calculation. Unlike in the case of the angular momentum, however, it is only the entire Hamiltonian which commutes with the Runge–Lenz vector, and *not* the kinetic and potential parts separately since

$$-\frac{\hbar^2}{2m_{\mathrm{e}}}[\nabla^2, R'_\mu] = e^2[r^{-1}, R'_\mu] = \frac{\hbar^2 e^2}{m_{\mathrm{e}}} \left[\frac{1}{r} \nabla_\mu - \frac{r_\mu}{r^3}(1 + \bar{r} \cdot \bar{\nabla}) \right] \neq 0. \tag{20}$$

Just as in the classical Kepler problem with its exceptional precessionless orbits, one finds that the commutator with the Runge–Lenz vector vanishes for a $1/r$ potential but not in general.

It is now established that both vectors $\bar{L}$ and $\bar{R}'$ commute with the Hamiltonian of the hydrogen atom and hence are constants of motion, but the symmetry of the system still needs to be determined. This can be done from the commutation relations among L_μ and R'_μ which read

$$[L_\mu, R'_\nu] = i\hbar \sum_{\rho=1}^{3} \epsilon_{\mu\nu\rho} R'_\rho, \qquad [R'_\mu, R'_\nu] = i\hbar \sum_{\rho=1}^{3} \epsilon_{\mu\nu\rho} \frac{-2H_{\mathrm{H}}}{m_{\mathrm{e}}} L_\rho, \tag{21}$$

together with the SO(3) relations among L_μ. Since the commutation relations among R'_μ do *not* give back L_ρ, one cannot claim that L_μ and R'_μ form a Lie algebra. In the space of eigenvectors corresponding to a single, negative eigenvalue, the following alternative operators can be introduced:

$$R_\mu = \sqrt{\frac{m_{\mathrm{e}}}{-2H_{\mathrm{H}}}} R'_\mu. \tag{22}$$

In general, the square-root of an operator has problematic properties but not in this case since it acts in a space of constant eigenvalue. Note also that one may rely here on the fact that neither L_μ nor R_μ or R'_μ can connect to states with a different energy eigenvalue, since they all commute with $H_{\rm H}$. The commutation relations among L_μ and R_μ now close,

$$[L_\mu, R_\nu] = i\hbar \sum_{\rho=1}^{3} \epsilon_{\mu\nu\rho} R_\rho, \qquad [R_\mu, R_\nu] = i\hbar \sum_{\rho=1}^{3} \epsilon_{\mu\nu\rho} L_\rho, \tag{23}$$

and the algebra consisting of L_μ and R_μ can be identified with SO(4), associated with the group of rotations in four dimensions. The relation (18) between the Hamiltonian and the conserved quantities $\bar{L}^2$ and $\bar{R}^2$ can be rewritten as

$$H_{\rm H} = -\frac{\hbar^2 R_{\rm H}}{\bar{L}^2 + \bar{R}^2 + \hbar^2}. \tag{24}$$

The operator occurring at the right-hand side of this identity, $\bar{L}^2 + \bar{R}^2$, can be identified with $C_2[\mathrm{SO}(4)]$, the quadratic Casimir operator of SO(4). The hydrogen atom provides thus a simple example in which a Hamiltonian can be written in terms of the Casimir operator of its symmetry algebra.

In general, if the symmetry group of a Hamiltonian is determined, its degeneracy structure follows automatically from the irreducible representations which can be looked up in monographs on group theory. In the case of SO(4) the analysis can be worked out with simple methods by converting to the operators

$$P_\mu = \frac{1}{2}\left(L_\mu + R_\mu\right), \qquad Q_\mu = \frac{1}{2}\left(L_\mu - R_\mu\right), \tag{25}$$

in terms of which the commutation relations become

$$[P_\mu, P_\nu] = i\hbar \sum_{\rho=1}^{3} \epsilon_{\mu\nu\rho} P_\rho, \quad [Q_\mu, Q_\nu] = i\hbar \sum_{\rho=1}^{3} \epsilon_{\mu\nu\rho} Q_\rho, \quad [P_\mu, Q_\nu] = 0. \tag{26}$$

The components P_μ commute with Q_ν and, furthermore, each set separately forms an SO(3) algebra. This, in fact, proves the isomorphism $\mathrm{SO}(4) \simeq \mathrm{SO}(3) \otimes \mathrm{SO}(3)$. Instead of relying on SO(4) representation theory, one can therefore use well-known results from SO(3). Since the operators P^2, P_z, Q^2 and Q_z commute with each other, and since they all commute with $H_{\rm H}$, they form a (complete) set of commuting operators. The eigenstates of $H_{\rm H}$ can then be labeled with the eigenvalues of the operators in this set and, in particular, with $p(p+1)\hbar^2$ and $q(q+1)\hbar^2$, the eigenvalues of the operators $\bar{P}^2$ and $\bar{Q}^2$. The allowed values of the labels

p and q are those of angular momentum, integer or half-integer, and for each value of p (q) there are $2p+1$ ($2q+1$) allowed substates. Furthermore, eigenstates of $H_{\rm H}$ necessarily have $p = q$ because the angular momentum and the Runge–Lenz vectors are orthogonal, $\bar{L} \cdot \bar{R} = 0$, which implies

$$\left(\bar{L} + \bar{R}\right)^2 = \left(\bar{L} - \bar{R}\right)^2 \Rightarrow \bar{P}^2 = \bar{Q}^2 \Rightarrow p(p+1) = q(q+1). \tag{27}$$

The allowed energy eigenvalues are now immediately obtained from (24) since the operator $\bar{L}^2 + \bar{R}^2 = 2\bar{P}^2 + 2\bar{Q}^2$ has the eigenvalue $4p(p+1)\hbar^2$,

$$E(p) = -\frac{\hbar^2 R_{\rm H}}{4p(p+1)\hbar^2 + \hbar^2} = -\frac{R_{\rm H}}{(2p+1)^2}, \qquad p = 0, \tfrac{1}{2}, 1, \ldots. \tag{28}$$

This coincides with the result (7) obtained from the standard quantum-mechanical derivation.

The hydrogen atom provides a beautiful application of symmetry. The degeneracies observed in the energy spectrum are higher than what is obtained from just rotational invariance. This requires the existence of a larger symmetry which is indeed found to be the case. Another illustration of this principle is provided by the spectrum of the harmonic oscillator in which case the underlying symmetry turns out to be U(3) [4].

A final comment concerns the method followed here to determine the eigenspectrum of the hydrogen atom. The standard way to do so is to solve the time-independent Schrödinger equation and to find the allowed values of the various quantum numbers from boundary conditions on the eigenfunctions. The procedure followed here is entirely different and exclusively based on the knowledge of a set of constants of motion which commute with the Hamiltonian, together with their mutual commutation relations. A crucial feature is that the Hamiltonian can be expressed in terms of the Casimir operator of the symmetry algebra. Although elegant and compact, the method itself does not provide an expression for the wave functions of stationary states. This ‘algebraic’ solution method of the problem of the hydrogen atom was proposed by Pauli in 1926 [5].

IV. DYNAMICAL SYMMETRY BREAKING

The concept of a dynamical symmetry for which (at least) two algebras G_1 and G_2 with $\mathrm{G}_1 \supset \mathrm{G}_2$ are needed can now be introduced. The eigenstates of a Hamiltonian H with symmetry G_1 are labeled as $|\Gamma_1\gamma_1\rangle$. But, since $\mathrm{G}_1 \supset \mathrm{G}_2$, a Hamiltonian with G_1 symmetry necessarily must also have a symmetry G_2 and, consequently, its eigenstates can also be labeled as $|\Gamma_2\gamma_2\rangle$. Combination of the two properties leads to the eigenequation

$$H|\Gamma_1\eta_{12}\Gamma_2\gamma_2\rangle = E(\Gamma_1)|\Gamma_1\eta_{12}\Gamma_2\gamma_2\rangle, \tag{29}$$

where the role of γ_1 is played by $\eta_{12}\Gamma_2\gamma_2$. The irreducible representation Γ_2 may occur more than once in Γ_1, and hence an additional quantum number η_{12} is needed to uniquely label the states. Because of G_1 symmetry, eigenvalues of H depend on Γ_1 only.

In many examples in physics (several are discussed below), the condition of G_1 symmetry is too strong and a *possible* breaking of the G_1 symmetry can be imposed via the Hamiltonian

$$H' = \kappa_1 C_{m_1}[\mathrm{G}_1] + \kappa_2 C_{m_2}[\mathrm{G}_2], \tag{30}$$

which consists of a combination of Casimir operators of G_1 *and* G_2. The symmetry properties of the Hamiltonian H' are now as follows. Since $[H', g_k] = 0$ for all g_k in G_2, H' is invariant under G_2. The Hamiltonian H', since it contains $C_{m_2}[\mathrm{G}_2]$, does not commute, in general, with all elements of G_1 and for this reason the G_1 symmetry is broken. Nevertheless, because H' is a combination of Casimir operators of G_1 and G_2, its eigenvalues can be obtained in closed form,

$$H'|\Gamma_1\eta_{12}\Gamma_2\gamma_2\rangle = [\kappa_1 E_{m_1}(\Gamma_1) + \kappa_2 E_{m_2}(\Gamma_2)]\,|\Gamma_1\eta_{12}\Gamma_2\gamma_2\rangle. \tag{31}$$

The conclusion is thus that, although H' is not invariant under G_1, its eigenstates are the same as those of H in (29). The Hamiltonian H' is said to have G_1 as a dynamical symmetry. The essential feature is that, although the eigenvalues of H' depend on Γ_1 *and* Γ_2 (and hence G_1 is not a symmetry), the eigenstates do not change during the breaking of the G_1 symmetry. As the generators of G_2 are a subset of those of G_1, the dynamical symmetry breaking splits but does not admix the eigenstates. A convenient way of summarizing the symmetry character of H' and the ensuing classification of its eigenstates is as follows:

$$\begin{array}{ccc} \mathrm{G}_1 & \supset & \mathrm{G}_2 \\ \downarrow & & \downarrow \\ \Gamma_1 & & \eta_{12}\Gamma_2 \end{array}\ . \tag{32}$$

This equation indicates the larger algebra G_1 (sometimes referred to as the dynamical algebra or spectrum generating algebra) and the symmetry algebra G_2, together with their associated labels with possible multiplicities.

Many concrete examples exist in physics of the abstract idea of dynamical symmetry. Perhaps the best known in nuclear physics concerns isospin symmetry and its breaking by the Coulomb interaction which is discussed in the next section.

V. ISOSPIN SYMMETRY

The starting point in the discussion of isospin symmetry is the observation that the masses of the neutron and proton are very similar, $m_{\rm n}c^2 = 939.55$ MeV and $m_{\rm p}c^2 = 938.26$ MeV, and that both have a spin of $\frac{1}{2}$. Furthermore, experiment shows that, if one neglects the contribution of the electromagnetic interaction, the forces between two neutrons are about the same as those between two protons. More precisely, the strong nuclear force between two nucleons with anti-parallel spins is found to be (approximately) independent of whether they are neutrons or protons. This indicates the existence of a symmetry of the strong interaction, and isospin is the appropriate formalism to explore the consequences of that symmetry in nuclei. The equality of the masses and the spins of the nucleons is not sufficient for isospin symmetry to be valid and the charge independence of the nuclear force is equally important. This point was emphasized by Wigner [6] who defined isospin for complex nuclei as we know it today and who also coined the name of 'isotopic spin'.

Because of the near-equality of the masses and of the interactions between nucleons, the Hamiltonian of the nucleus is (approximately) invariant with respect to transformations between neutron and proton states. For one nucleon, these can be defined by introducing the abstract space spanned by the two vectors

$$|{\rm n}\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |{\rm p}\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{33}$$

The most general transformation among these states (which conserves their normalization) is a unitary 2×2 matrix. A matrix close to the identity can be represented as

$$\begin{bmatrix} 1+\epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & 1+\epsilon_{22} \end{bmatrix}, \tag{34}$$

where the ϵ_{ij} are infinitesimal complex numbers. Unitarity imposes the relations

$$\epsilon_{11} + \epsilon_{11}^* = \epsilon_{22} + \epsilon_{22}^* = \epsilon_{12} + \epsilon_{21}^* = 0. \tag{35}$$

An additional condition is found by requiring the determinant of the unitary matrix to be equal to +1,

$$\epsilon_{11} + \epsilon_{22} = 0, \tag{36}$$

which removes the freedom to make a simultaneous and identical change of phase for the neutron and the proton. The infinitesimal, physical transformations between a neutron and a proton can therefore be parametrized as

$$\begin{bmatrix} 1 - \frac{1}{2}i\epsilon_z & -\frac{1}{2}i(\epsilon_x - i\epsilon_y) \\ -\frac{1}{2}i(\epsilon_x + i\epsilon_y) & 1 + \frac{1}{2}i\epsilon_z \end{bmatrix}, \tag{37}$$

which includes a conventional factor $-i/2$ and where the $\{\epsilon_x, \epsilon_y, \epsilon_z\}$ now are infinitesimal real numbers. This can be rewritten in terms of the Pauli spin matrices as

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{2}i\epsilon_x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \frac{1}{2}i\epsilon_y \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \frac{1}{2}i\epsilon_z \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{38}$$

The infinitesimal transformations between a neutron and a proton can thus be written in terms of the three operators

$$t_x \equiv \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad t_y \equiv \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad t_z \equiv \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{39}$$

which satisfy *exactly* the same commutation relations as the angular momentum operators. The action of the t_μ operators on a nucleon state is easily found from its matrix representation. For example,

$$t_z|\mathrm{n}\rangle \equiv \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}|\mathrm{n}\rangle, \quad t_z|\mathrm{p}\rangle \equiv \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{1}{2}|\mathrm{p}\rangle, \tag{40}$$

which shows that $e(1 - 2t_z)/2$ is the charge operator. Also, the combinations $t_\pm \equiv t_x \pm it_y$ can be introduced, which satisfy the commutation relations

$$[t_z, t_\pm] = \pm t_\pm, \qquad [t_+, t_-] = 2t_z, \tag{41}$$

and play the role of raising and lowering operators since

$$t_-|\mathrm{n}\rangle = |\mathrm{p}\rangle, \qquad t_+|\mathrm{n}\rangle = 0, \qquad t_-|\mathrm{p}\rangle = 0, \qquad t_+|\mathrm{p}\rangle = |\mathrm{n}\rangle. \tag{42}$$

This proves the formal equivalence between spin and isospin, and all results familiar from angular momentum can now be readily transposed to the isospin algebra. For a many-nucleon system (such as a nucleus) a total isospin T and its z projection M_T can be defined which results from the coupling of the individual isospins, just as this can be done for the nucleon spins. The appropriate isospin operators are

$$T_\mu = \sum_{k=1}^{A} t_\mu(k), \tag{43}$$

where the sum is over all the nucleons in the nucleus.

If, in first approximation, the Coulomb interaction between the protons is neglected and, furthermore, if it is assumed that the strong interaction does not distinguish between neutrons and protons, the resulting nuclear Hamiltonian H is isospin invariant. Explicitly, invariance under the isospin algebra $\mathrm{SU}(2) \equiv \{T_z, T_\pm\}$ follows from

$$[H, T_z] = [H, T_\pm] = 0. \tag{44}$$

As a consequence of these commutation relations, the many-particle eigenstates of H have good isospin symmetry. They can be classified as $|\eta T M_T\rangle$ where T is the total isospin of the nucleus obtained from the coupling of the individual isospins $\frac{1}{2}$ of all nucleons, M_T is its projection on the z axis in isospin space, $M_T = (N-Z)/2$ and η denotes all additional quantum numbers. If isospin were a true symmetry, all states $|\eta T M_T\rangle$ with $M_T = -T, -T+1, \ldots, +T$, and with the same T (and identical other quantum numbers η), would be degenerate in energy; for example, neutron and proton would have exactly the same mass. States with the same ηT but different M_T (and hence in different nuclei) are referred to as isobaric analogue states.

A. The isobaric multiplet mass equation

The Coulomb interaction between the protons destroys the equivalence between the nucleons and hence breaks isospin symmetry. The main effect of the Coulomb interaction is a *dynamical* breaking of isospin symmetry. This can be shown by rewriting the Coulomb interaction,

$$V_\mathrm{C} = \sum_{k<l}^{A} \left(\frac{1}{2} - t_z(k)\right)\left(\frac{1}{2} - t_z(l)\right) \frac{e^2}{|\bar{r}_k - \bar{r}_l|}, \tag{45}$$

as a sum of isoscalar, isovector and isotensor parts

$$V_\mathrm{C} = \sum_{k<l}^{A} \sum_{t=0,1,2} V_0^{(t)}(k,l), \tag{46}$$

with

$$
\begin{aligned}
V_0^{(0)}(k,l) &= \left(\frac{1}{4}-\sqrt{\frac{1}{3}}\,[\bar{t}(k)\times\bar{t}(l)]_0^{(0)}\right)\frac{e^2}{|\bar{r}_k-\bar{r}_l|},\\
V_0^{(1)}(k,l) &= -\frac{1}{2}\,[t_z(k)+t_z(l)]\,\frac{e^2}{|\bar{r}_k-\bar{r}_l|},\\
V_0^{(2)}(k,l) &= \sqrt{\frac{2}{3}}\,[\bar{t}(k)\times\bar{t}(l)]_0^{(2)}\,\frac{e^2}{|\bar{r}_k-\bar{r}_l|},
\end{aligned}
\tag{47}
$$

where the coupling is carried out in isospin. The Wigner–Eckart theorem in isospin space allows to factor out the M_T dependence of any diagonal matrix element according to

$$
\langle\eta TM_T|\sum_{k<l}^{A}V_0^{(t)}(k,l)|\eta TM_T\rangle = \langle TM_T\; t0|TM_T\rangle\langle\eta T\|\sum_{k<l}^{A}V^{(t)}(k,l)\|\eta T\rangle, \tag{48}
$$

where $\langle TM_T\; t0|TM_T\rangle$ is a Clebsch–Gordan coefficient associated with SU(2) $\supset$ SO(2). From the explicit expressions for these coefficients,

$$
\begin{aligned}
\langle TM_T\; 00|TM_T\rangle &= 1, \qquad \langle TM_T\; 10|TM_T\rangle = \frac{M_T}{\sqrt{T(T+1)}},\\
\langle TM_T\; 20|TM_T\rangle &= \frac{3M_T^2-T(T+1)}{\sqrt{T(T+1)(2T-1)(2T+3)}},
\end{aligned}
\tag{49}
$$

one concludes that the M_T dependence of the diagonal matrix elements of the Coulomb interaction is at most quadratic. If the off-diagonal, isospin mixing matrix elements of V_{C} are neglected, it can then be represented as

$$
V_{\mathrm{C}} \approx \kappa_0+\kappa_1 T_z+\kappa_2 T_z^2, \tag{50}
$$

for some particular coefficients κ_0, κ_1 and κ_2 which, according to the preceding discussion, depend on the isospin T and other quantum numbers η. This can be viewed as a dynamical symmetry breaking of the type

$$
\begin{array}{ccc}
\mathrm{SU}(2) & \supset & \mathrm{SO}(2)\equiv\{T_z\}\\
\downarrow & & \downarrow\\
T & & M_T
\end{array}
. \tag{51}
$$

The Hamiltonian (50) splits but does not admix the eigenstates $|\eta TM_T\rangle$ with $M_T = -T, -T+1, \ldots, +T$, and has the eigenspectrum

$$
E(M_T) = \kappa_0+\kappa_1 M_T+\kappa_2 M_T^2. \tag{52}
$$

The expansion in T_z is but an approximation to the true Coulomb interaction; it represents the diagonal part of it, with the T-mixing isovector and isotensor parts being neglected. In that approximation isospin remains a good quantum number. The excitation spectra of the different nuclei belonging to the same isospin multiplet (with the same T but different M_T) are identical but their ground states do not have the same binding energy. The energy formula in M_T was derived by Wigner [7] who introduced the name of isobaric multiplet mass equation (IMME). Many experimental examples of nuclear isospin multiplets are known at present [8].

The assumption of isospin symmetry is too strong and should be relaxed to one of dynamical symmetry. One cannot expect that isobaric analogue states have the same *absolute* energy but one can expect them to have, to a good approximation, the same *relative* energies. As a result, for example, the excitation spectra of two mirror nuclei should be identical although the binding energy of their ground states differs. (Mirror nuclei have the same total number of nucleons and the number of neutrons in one of them equals the number of protons in the other.) This relation has been observed in many cases. An example where the idea has been tested to high angular momentum, is shown in Fig. 3 [9]. The ground-state energies of the two nuclei of the $T = \frac{1}{2}$ isospin doublet (^{49}Cr with $M_T = +\frac{1}{2}$ and ^{49}Mn with $M_T = -\frac{1}{2}$) are shifted with respect to each other but the energies relative to the ground state are indeed very similar. Nevertheless, the spectra are not identical as is clear from the inset in Fig. 3 where the difference in excitation energy is plotted as a function of the angular momentum J. The deviations from zero signal a breakdown of the dynamical-symmetry approximation and, specifically, reveal subtle differences in alignment properties of the neutrons and protons in the two mirror nuclei [10].

The equality of excitation spectra of mirror nuclei is sometimes referred to as mirror symmetry. It should be emphasized that mirror symmetry is but a particular manifestation of isospin symmetry which implies a wider relationship between properties of nuclei as illustrated with the example in Fig. 4. The nuclei shown contain $A = 14$ nucleons but differ by their numbers of neutrons and protons, $(N, Z) = (8, 6)$, (7,7) and (6,8). This corresponds to eigenvalues of T_z given by $M_T = (N - Z)/2 = +1, 0, -1$ and, consequently, the isospin of all states in ^{14}C and ^{14}O must be $T = 1$ or higher. As a consequence of mirror symmetry, the low-energy spectra of both nuclei should be identical. The $T = 1$ analogue states should also occur in ^{14}N, however. This nucleus has $M_T = 0$ but this does not preclude the existence

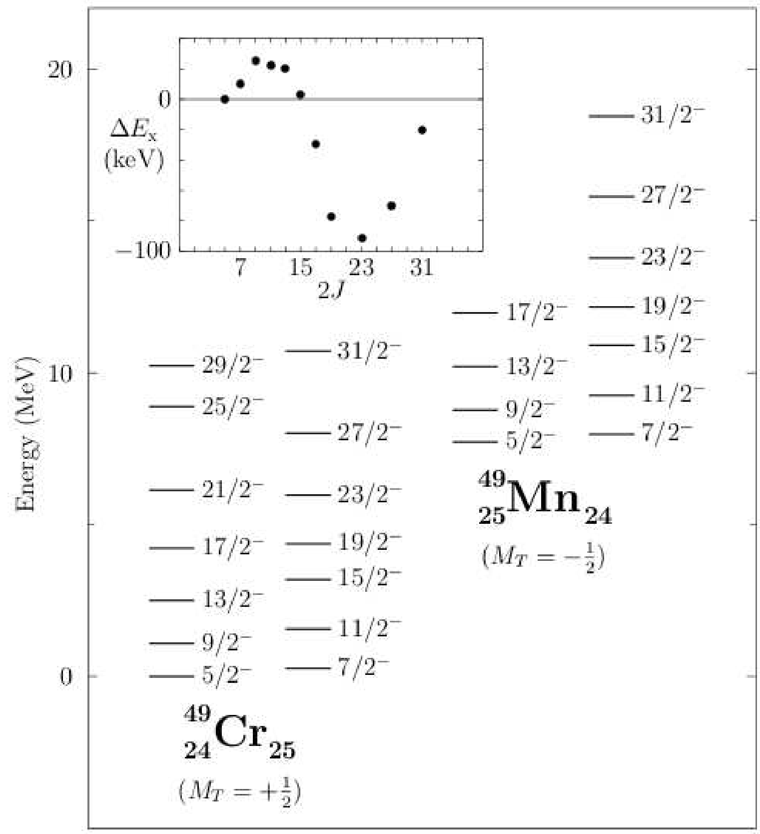

FIG. 3: Energy spectra of the mirror nuclei ^{49}Cr and ^{49}Mn relative to the ground state of the first nucleus. Levels are labeled by their angular momentum and parity J^π. The inset shows the difference in excitation energy $\Delta E_x \equiv E_x(^{49}\mathrm{Cr}; J) - E_x(^{49}\mathrm{Mn}; J)$ as a function of $2J$.

of $T = 1$ states. In fact, isospin symmetry *requires* that such states be present somewhere in the spectrum of ^{14}N. Figure 4 illustrates that the isobaric analogue levels of those in ^{14}C and ^{14}O are indeed found in ^{14}N.

For $T \geq \frac{3}{2}$ it is possible to test the IMME since the parameters κ_i can be fixed from the isobaric analogue states in three nuclei and a prediction follows for the fourth member of the multiplet. As an example consider the $T = \frac{3}{2}$ multiplet consisting of isobaric analogue states in ^{13}B, ^{13}C, ^{13}N and ^{13}O. Figure 5 shows the binding energies of the nuclei ^{13}B and ^{13}O, both of which have $T = |M_T| = \frac{3}{2}$ in their ground state. The isobaric analogue states in ^{13}C and ^{13}N are $J^\pi = \frac{1}{2}^-$ states at excitation energies of 15.11 and 15.07 MeV, respectively; these energies are substracted from the ground-state binding energies of ^{13}C and ^{13}N to give the energies plotted in Fig. 5. In this example the energy splitting due to the Coulomb interaction is well accounted for by the IMME, which is perhaps not surprising since four

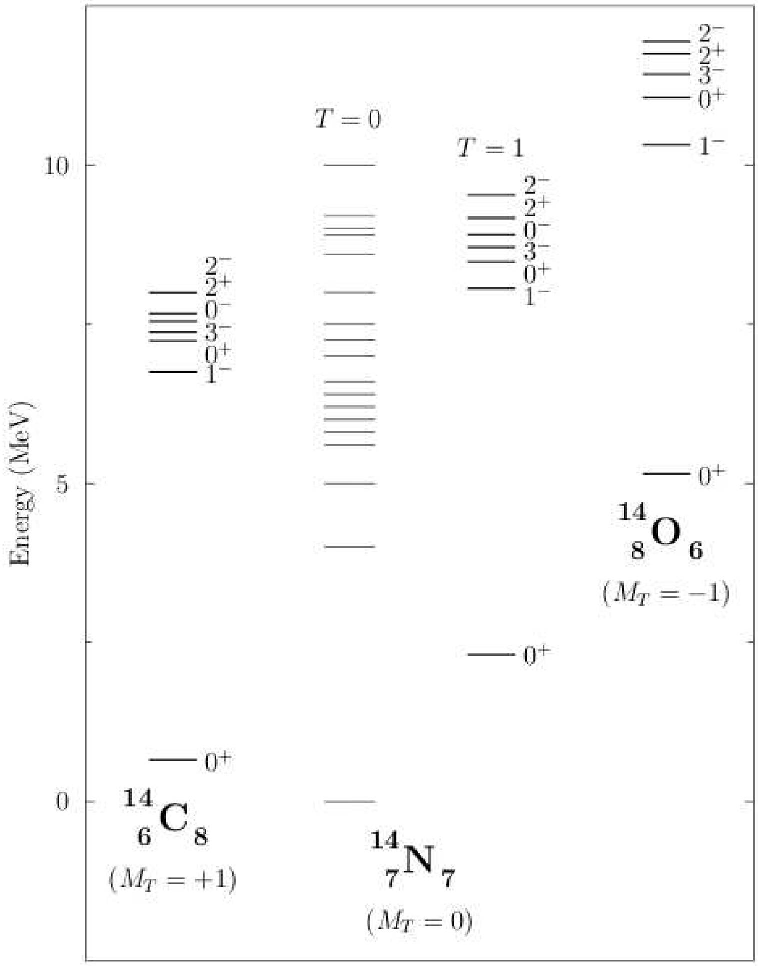

FIG. 4: Energy spectra of the nuclei ^{14}C, ^{14}N and ^{14}O relative to the ground state of the middle nucleus. States with isospin $T = 1$ are drawn in thick lines. In the self-conjugate nucleus ^{14}N there exist also states with isospin $T = 0$ which are drawn in thin lines. Levels with $T = 1$ are labeled by their angular momentum and parity J^π.

data points are fitted with three parameters. The quality of fits such as the one in Fig. 5 is, however, not the most important aspect of dynamical symmetries, but rather the existence of good quantum numbers (isospin T in this case).

B. Isospin selection rules

The most important consequence of a symmetry, which remains valid under the process of a dynamical symmetry breaking, is the existence of conserved (or 'good') quantum numbers. Frequently, these quantum numbers give rise to selection rules in radiative transition or

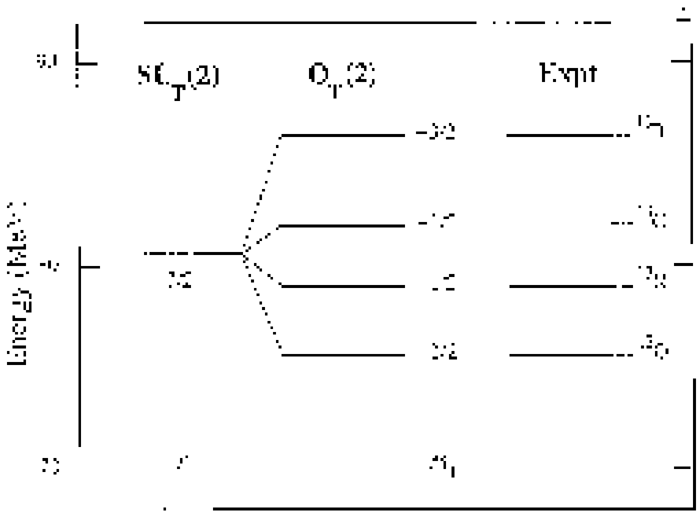

FIG. 5: Binding energies of the $T = \frac{3}{2}$ isobaric analogue states with $J^\pi = \frac{1}{2}^-$ in ^{13}B, ^{13}C, ^{13}N and ^{13}O. The column on the left is obtained for an exact $\mathrm{SU}_T(2)$ isospin symmetry, which predicts states with different M_T to be degenerate. The middle column is obtained with the IMME with $\kappa_0 = 80.59$, $\kappa_1 = -2.96$ and $\kappa_2 = -0.26$, in MeV.

particle-transfer processes. The measurement of transition or transfer probabilities is thus the method to establish the goodness of labels needed to characterize a quantum state and this in turn indicates to what extent a given (dynamical) symmetry is valid.

The link between symmetries and selection rules can be given a precise quantitative formulation via the (generalized) Wigner–Eckart theorem. Suppose the calculation is required of a transition or transfer matrix element between an initial state $|\Gamma_\mathrm{i}\gamma_\mathrm{i}\rangle$ and a final state $|\Gamma_\mathrm{f}\gamma_\mathrm{f}\rangle$, where the labeling of Subsect. II B is adopted. To compute the matrix element, it is first necessary to determine the tensor character of the operator associated with the transition or transfer by formally writing the operator as $\sum_{\Gamma\gamma} a_{\Gamma\gamma} T^\Gamma_\gamma$ where $a_{\Gamma\gamma}$ are coefficients. Each piece T^Γ_γ can now be dealt with separately through the generalized Wigner–Eckart theorem. The essential point is that all dependence on the quantum numbers associated with the subalgebra G_2 is contained in a generalized coupling coefficient. In addition, selection rules now follow from the multiplication rules for irreducible representations of the algebra G_1: if Γ_f is not contained in the product $\Gamma_\mathrm{i} \times \Gamma$, the generalized coupling coefficient is zero and the matrix element of T^Γ_γ vanishes.

A well-known example of the idea of selection rules concerns electric dipole transitions in self-conjugate nuclei [11, 12], that is, nuclei with an equal number of neutrons and protons ($N = Z$). The E1 operator is, in lowest order of the long-wave approximation, given by

$$T_\mu(\mathrm{E1}) = \sum_{k=1}^{A} e_k r_\mu(k). \tag{53}$$

Since the charge e_k of the k^{th} nucleon is zero for a neutron and e for a proton, the E1 operator can be rewritten as

$$T_\mu(\mathrm{E1}) = \frac{e}{2}\sum_{k=1}^{A}[1 - 2t_z(k)]r_\mu(k) = \frac{e}{2}\left[R_\mu - 2\sum_{k=1}^{A} t_z(k)r_\mu(k)\right], \tag{54}$$

where $2t_z$ gives $+1$ for a neutron and -1 for a proton. The first term R_μ in the E1 operator is the centre-of-mass coordinate of the total nucleus and does not contribute to an internal E1 transition. The conclusion is that the electric dipole operator is, in lowest order of the long-wave approximation, of pure isovector character. The application of the Wigner–Eckart theorem in isospin space gives

$$\langle \eta_\mathrm{f} T_\mathrm{f} M_{T_\mathrm{f}} | T^{(1)}_0 | \eta_\mathrm{i} T_\mathrm{i} M_{T_\mathrm{i}} \rangle = \langle T_\mathrm{i} M_{T_\mathrm{i}}\ 10 | T_\mathrm{f} M_{T_\mathrm{f}} \rangle \langle \eta_\mathrm{f} T_\mathrm{f} \| T^{(1)} \| \eta_\mathrm{i} T_\mathrm{i} \rangle, \tag{55}$$

where the coupling coefficient is associated with SU(2) $\supset$ SO(2). Self-conjugate nuclei have $M_{T_\mathrm{i}} = M_{T_\mathrm{f}} = 0$ and exhibit as a consequence a simple selection rule: E1 transitions are forbidden between levels with the same isospin $T_\mathrm{i} = T_\mathrm{f} = T$ because of the vanishing Clebsch–Gordan coefficient, $\langle T0\ 10|T0\rangle = 0$.

This selection rule has been verified to hold approximately in light self-conjugate nuclei [13]. Deviations occur because of higher-order terms in the E1 operator but also, and more importantly, because isospin is not an exactly conserved quantum number. Isospin mixing can be estimated in a variety of nuclear models. They all show that the mixing (*i.e.*, the non-dynamical breaking of isospin symmetry) is maximal in $N = Z$ nuclei. Isospin mixing effects, caused mainly by the Coulomb interaction, should thus be looked for in heavy $N = Z$ nuclei where they are largest. Such nuclei are created and accelerated for study at radioactive-ion beam facilities. The spectrum of an $N = Z$ nucleus studied in this respect, ^{64}Ge, is shown in Fig. 6. The crucial transition is the E1 between the 5^- and 4^+ levels (indicated by the down arrow) which should be strictly forbidden if the isospin dynamical symmetry were exact. A small $B(\mathrm{E1}; 5^- \to 4^+)$ value is measured nevertheless and this is explained through the mixing with higher-lying 5^- and 4^+ levels in ^{64}Ge with $T = 1$,

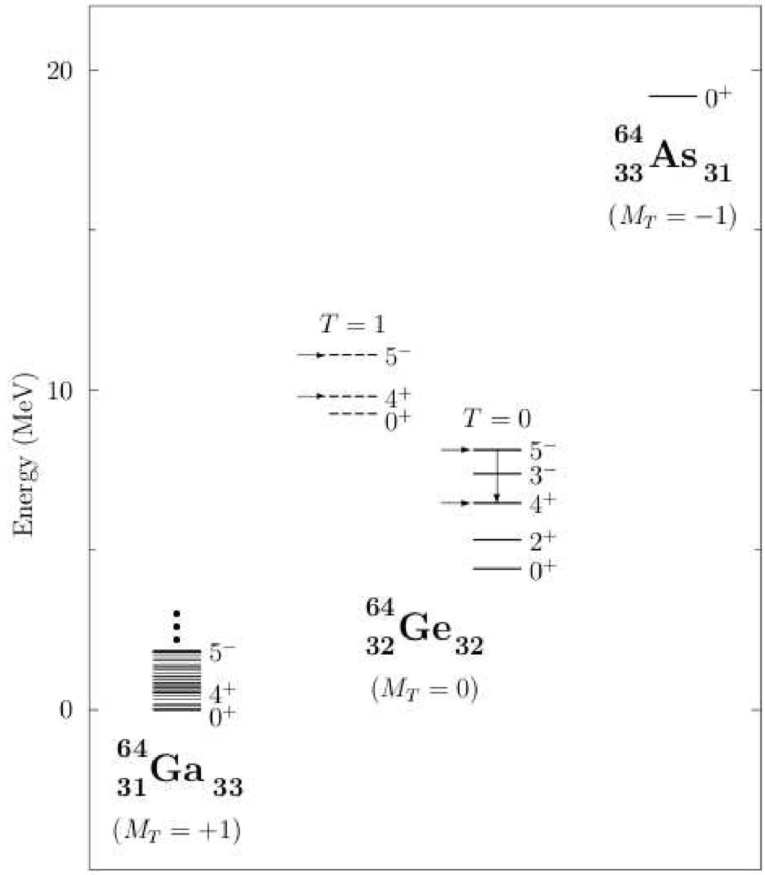

FIG. 6: Energy spectra of the nuclei in the $A = 64$ isospin triplet ^{64}Ga, ^{64}Ge and ^{64}As relative to the ground state of the first nucleus. The observed $5^- \rightarrow 4^+$ E1 transition between $T = 0$ states in ^{64}Ge is explained through mixing with the $T = 1$ states, indicated by the arrows. The levels in broken lines are inferred from the isospin analogue levels in ^{64}Ga.

which are not observed but inferred from their isospin analogue states in ^{64}Ga. Although an estimate of the isospin mixing can be made in this way, the procedure is difficult as it requires the measurement of the lifetime, the δ(E2/M1) mixing ratio and the relative intensities of the transitions de-exciting the 5^- level [14]. Given these uncertainties, a reliable measurement of isospin admixtures in nuclei, as a function of N and Z, is still a declared goal of the current experimental efforts with radioactive-ion beams.

VI. THE NUCLEAR SHELL MODEL

The structure of the atomic nucleus is determined, in first approximation, by the nuclear mean field, the average potential felt by a nucleon through the interactions exerted by all others. This average potential is responsible for the shell structure of the nucleus because the energy spectrum of a particle moving in this mean field shows regions with many levels and others with few. A second important ingredient that determines the structure of nuclei (and generally of many-body quantum systems) is the Pauli principle. Consequently, the nucleus can be viewed as an onion-like construction, with shells determined by the mean-field potential that are being filled in accordance with the Pauli principle. For a description that goes beyond this most basic level, the residual interaction between nucleons must be taken into account and what usually matters most for nuclear structure at low energies is the residual interaction between nucleons in the valence or outer shell. This interaction depends in a complex fashion on the numbers of valence neutrons and protons, and on the valence orbits available to them.

No review is given here of the nuclear shell model which has been the subject of several comprehensive monographs [15–19]. Instead, after an introductory subsection, describing the model's essential features and assumptions, emphasis is laid on its symmetry structure. It turns out that the two most important correlations in nuclei, of the pairing and of the quadrupole type, respectively, can be analyzed with symmetry techniques.

A. The model

In a non-relativistic approximation, the wavefunction of any quantum-mechanical state of a nucleus with A nucleons satisfies the Schrödinger equation

$$H\Psi(\xi_1,\xi_2,\ldots,\xi_A) = E\Psi(\xi_1,\xi_2,\ldots,\xi_A), \tag{56}$$

with the Hamiltonian

$$H = \sum_{k=1}^{A} \frac{p_k^2}{2m_k} + \sum_{k<l}^{A} W_2(\xi_k,\xi_l) + \sum_{k<l<m}^{A} W_3(\xi_k,\xi_l,\xi_m) + \cdots. \tag{57}$$

The notation ξ_k is used to denote all coordinates of nucleon k, not only its position vector $\bar{r}_k$ but also its spin $\bar{s}_k$ and its isospin $\bar{t}_k$, $\xi_k \equiv \{\bar{r}_k, \bar{s}_k, \bar{t}_k\}$. The term $p_k^2/2m_k$ is the kinetic energy of nucleon k and acts on a single nucleon only. The operator $W_i(\xi_k, \xi_l, \xi_m, \ldots)$ is an i-body interaction between the nucleons $k, l, m, \ldots$, and, as such, acts on i nucleons simultaneously. Since neutron and proton are not elementary particles, it is not *a priori* clear that the interaction should be of two-body nature. Nevertheless, for a presentation of the elementary nuclear shell model, it can be assumed that the nature between the nucleons is two-body, $W_{i>2} = 0$, as will be done in the subsequent discussion.

Under the assumption of at most two-body interactions, one can rewrite (57) as

$$H = \sum_{k=1}^{A} \left(\frac{p_k^2}{2m_k} + V(\xi_k) \right) + \left(\sum_{k<l}^{A} W_2(\xi_k, \xi_l) - \sum_{k=1}^{A} V(\xi_k) \right). \tag{58}$$

The idea is now to choose $V(\xi_k)$ such that the effect of the residual interaction, that is, the second term in (58), is minimized. The independent-particle shell model is obtained by neglecting the residual interaction altogether,

$$H_{\mathrm{ip}} = \sum_{k=1}^{A} \left(\frac{p_k^2}{2m_{\mathrm{n}}} + V(\xi_k) \right), \tag{59}$$

where it is also assumed that all nucleons have the same mass m_{n}. The physical interpretation of the approximation (59) is that each nucleon moves independently in a mean-field potential $V(\xi)$ which represents the average interaction with all other nucleons in the nucleus.

The eigenproblem associated with the Hamiltonian (59) is much easier to solve than the original problem (56) because it can be reduced to a one-particle eigenequation. Its solution proceeds as follows. First, one solves the Schrödinger equation of a particle in a potential $V(\xi)$, that is, one finds the eigenfunctions $\phi_i(\xi)$ satisfying

$$\left(\frac{p^2}{2m_{\mathrm{n}}} + V(\xi) \right) \phi_i(\xi) = E_i \phi_i(\xi), \tag{60}$$

where i labels the different eigensolutions. The exact form of the eigenfunctions $\phi_i(\xi)$ depends on the potential $V(\xi)$. For simple potentials (*e.g.*, the harmonic oscillator) the eigenfunctions can be found in analytic form in terms of standard mathematical functions; for more complicated potentials (*e.g.*, Woods–Saxon) $\phi_i(\xi)$ must be determined numerically. For all 'reasonable' potentials $V(\xi)$ the solutions of (60) can be obtained, albeit in most cases only in numerical form.

The solution of the *many-body* Hamiltonian $H_{\rm ip}$ is immediately obtained due to its separability,

$$\Phi_{i_1 i_2 \ldots i_A}(\xi_1, \xi_2, \ldots, \xi_A) = \prod_{k=1}^{A} \phi_{i_k}(\xi_k). \tag{61}$$

Although this is a genuine, mathematical eigensolution of the Hamiltonian (59), it is not antisymmetric under the exchange of particles as is required by the Pauli principle. The solution (61) must thus be antisymmetrized. For $A = 2$ particles the antisymmetrization procedure yields

$$\begin{aligned} \Psi_{i_1 i_2}(\xi_1, \xi_2) &= \sqrt{\frac{1}{2}}[\phi_{i_1}(\xi_1)\phi_{i_2}(\xi_2) - \phi_{i_1}(\xi_2)\phi_{i_2}(\xi_1)] \\ &= \sqrt{\frac{1}{2}} \begin{vmatrix} \phi_{i_1}(\xi_1) & \phi_{i_1}(\xi_2) \\ \phi_{i_2}(\xi_1) & \phi_{i_2}(\xi_2) \end{vmatrix}. \end{aligned} \tag{62}$$

In the A-particle case, antisymmetrization leads to the replacement of the wave function $\Phi_{i_1 i_2 \ldots i_A}(\xi_1, \xi_2, \ldots, \xi_A)$ by a Slater determinant of the form

$$\Psi_{i_1 i_2 \ldots i_A}(\xi_1, \xi_2, \ldots, \xi_A) = \frac{1}{\sqrt{A!}} \begin{vmatrix} \phi_{i_1}(\xi_1) & \phi_{i_1}(\xi_2) & \cdots & \phi_{i_1}(\xi_A) \\ \phi_{i_2}(\xi_1) & \phi_{i_2}(\xi_2) & \cdots & \phi_{i_2}(\xi_A) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{i_A}(\xi_1) & \phi_{i_A}(\xi_2) & \cdots & \phi_{i_A}(\xi_A) \end{vmatrix}. \tag{63}$$

This is the solution of the Schrödinger equation associated with the Hamiltonian (59) that takes account of the Pauli principle.

The following question now arises. How should one choose the potential $V(\xi)$ introduced in (58)? This choice can be made at several levels of refinement. Ideally one wants to minimize the expectation value of H in the ground state, that is, solve the variational equation

$$\delta \int \Psi^*(\xi_1, \xi_2, \ldots, \xi_A) H \Psi(\xi_1, \xi_2, \ldots, \xi_A) d\xi_1 d\xi_2 \ldots d\xi_A = 0. \tag{64}$$

If, in this variational approach, the wave function $\Psi(\xi_1, \xi_2, \ldots, \xi_A)$ is allowed to vary freely, the solution of (64) is equivalent to the ground-state solution of the Schrödinger equation (56). Obviously, one needs to set more modest goals to arrive at a solvable problem! One way to do so is to restrict $\Psi(\xi_1, \xi_2, \ldots, \xi_A)$ in (64) to the form of a Slater determinant, in other words, to minimize the ground-state energy by varying the potential $V(\xi)$ that defines the single-particle wave functions $\phi_{i_1}, \phi_{i_2}, \ldots, \phi_{i_A}$ in (63). This is known as the Hartree–Fock method. One determines the form of the potential $V(\xi)$ by requiring the expectation value of the *complete* Hamiltonian (57) in the state (63) to be minimal.

The ground-state energy determined in Hartree–Fock theory is not the correct one; nevertheless, it is the best procedure at hand to construct an independent-particle model. Its disadvantage is that it can be computationally rather involved. Therefore, often the following simpler approach is preferred. One proposes a phenomenological form of the potential $V(\xi)$, such that the Schrödinger equation associated with H_{ip} in (59) is analytically solvable. The potential which best mimics the nuclear mean-field potential and which can be solved exactly, is the harmonic-oscillator potential

$$V(\xi) \equiv V(r) = \frac{1}{2} m_{\mathrm{n}} \omega^2 r^2. \tag{65}$$

The eigensolutions of the Schrödinger equation of a harmonic oscillator in three dimensions can be written as

$$\phi_{n\ell m_\ell}(r,\theta,\varphi) = R_{n\ell}(r) Y_{\ell m_\ell}(\theta,\varphi), \tag{66}$$

where $R_{n\ell}(r)$ are radial wave functions appropriate for the harmonic oscillator and $Y_{\ell m_\ell}(\theta,\varphi)$ are spherical harmonics, already introduced in Eq. (8). The index i, used previously to characterize single-particle eigenfunctions, is replaced now by the full set of quantum numbers n, ℓ and m_ℓ. The quantized energy spectrum is given by

$$E(n,\ell) = \left(2n + \ell + \tfrac{3}{2}\right) \hbar\omega, \tag{67}$$

in terms of the radial quantum number n which has the allowed values $0, 1, 2, \ldots$ and gives the number of nodes [values of r for which $R_{n\ell}(r) = 0$ excluding those at $r = 0$ and $r = \infty$]. Because of the factor r^ℓ in the radial part, the wave function always vanishes at $r = 0$ except for $\ell = 0$ (s state). The energy $E(n,\ell)$ is independent of m_ℓ, the projection of the orbital angular momentum along the z axis, as should be for a rotationally invariant Hamiltonian. In addition, $E(n,\ell)$ is only dependent on the sum $2n+\ell$. Introducing $N = 2n + \ell$, one can rewrite (67) as

$$E(N) = \left(N + \tfrac{3}{2}\right) \hbar\omega, \tag{68}$$

which shows that N can be interpreted as the number of oscillator quanta, the term $\frac{3}{2}\hbar\omega$ being accounted for by the zero-point motion of an oscillator in three dimensions; N is called the major oscillator quantum number. The allowed values of the orbital angular momentum are (because $\ell = N - 2n$ and $n = 0, 1, \ldots$)

$$\ell = N, N - 2, \ldots, 0 \text{ or } 1. \tag{69}$$

This completely determines the eigenspectrum of a spinless particle in a harmonic-oscillator potential.

The $2\ell + 1$ eigensolutions with the same radial quantum number n and the same orbital angular momentum ℓ but different z projections m_ℓ are degenerate in energy. This degeneracy arises because the harmonic-oscillator Hamiltonian is rotationally invariant. There exists an *additional* degeneracy, namely the one for levels with the same $2n + \ell$. As in the case of the spectrum of the hydrogen atom, discussed in Sect. III, this additional degeneracy is also associated with a symmetry of the Hamiltonian which is identified in this case as U(3) [4]. The U(3) transformations are more general than rotations in three dimensions [*i.e.*, U(3) contains SO(3)] and U(3) invariance can be understood intuitively as a consequence of the equivalence between the excitation of quanta in the x, y and z directions. The degeneracies of the harmonic-oscillator energy levels do not occur for a Woods–Saxon potential. In general one finds for a Woods–Saxon potential that, of the orbits with the same major oscillator quantum number N, those with high ℓ are more strongly bound than those with low orbital angular momentum.

An important quantity appearing in the harmonic-oscillator model is the elementary quantum of excitation $\hbar\omega$. By relating the radius of the nucleus, R, to the number of nucleons, A, and subsequently deriving a relationship between R, A and the oscillator length b, one finds the expression [16]

$$b \approx 1.00 A^{1/6} \text{ fm}, \tag{70}$$

and, since $b = \sqrt{\hbar / m_\mathrm{n} \omega}$,

$$\hbar\omega \approx 41 A^{-1/3} \text{ MeV}. \tag{71}$$

The solutions $\phi_{n\ell m_\ell}(r, \theta, \varphi)$ contain the dependence on the spatial coordinates only and not on the intrinsic spin of the particle. Since the intrinsic spin does not appear in the potential (65), the wave functions are simply given by

$$\phi_{n\ell m_\ell}(r, \theta, \varphi) \chi_{s m_s}, \tag{72}$$

where $\chi_{s m_s}$ are spinors for particles with intrinsic spin $s = \frac{1}{2}$. The energies are independent of m_s and are still given by (67). The eigenstates (72) do not have good *total* angular momentum, that is, they are not eigenstates of j^2 where $\bar{j}$ results from the coupling of the orbital angular momentum $\bar{\ell}$ and the spin $\bar{s}$ of the nucleon. States of good angular momentum are constructed from (72) with the help of Clebsch–Gordan coefficients,

$$\phi_{n\ell jm_j}(r,\theta,\varphi) = \sum_{m_\ell m_s} (\ell m_\ell\, s m_s|jm_j)\phi_{n\ell m_\ell}(r,\theta,\varphi)\chi_{sm_s}. \tag{73}$$

Again, this state has the same energy eigenvalue (67) since all states appearing in the sum are degenerate.

If the spin degeneracy of the quantum numbers $(n\ell jm_j)$ is taken into account, stable shell gaps are obtained at the nucleon numbers 2, 8, 20, 40, 70, 112,.... These are the magic numbers of the harmonic oscillator.

The existence of nuclear shell structure can be demonstrated in a variety of ways. The most direct way is by measuring the ease with which a nucleus can be excited. If it has a closed shell structure, one expects it to be rather stable and difficult to excite. This should be particularly so for nuclei that are doubly magic, that is, nuclei with a closed-shell configuration for neutrons *and* protons. The principle is illustrated in Fig. 7. The figure shows the energy $E_{\rm x}(2_1^+)$ of the first-excited 2^+ state relative to the ground state for all even–even nuclei. This energy is multiplied with $A^{1/3}$ and the result plotted on a normalized scale. (The factor $A^{1/3}$ accounts for the gradual decrease with mass number A of the strength of the nuclear residual interaction which leads a compression of the spectrum with A.) Nuclei with particularly high values of $E_{\rm x}(2_1^+)A^{1/3}$ are ^{16}O ($N = Z = 8$), ^{40}Ca ($N = Z = 20$), ^{48}Ca ($N = 28$, $Z = 20$), ^{132}Sn ($N = 82$, $Z = 50$) and ^{208}Pb ($N = 126$, $Z = 82$). Figure 7 establishes the stability properties of the isotopes and/or isotones with $N, Z = 8$, 20, 28, 50, 82 and 126.

How to explain the differences between the observed magic numbers (2, 8, 20, 28, 50, 82 and 126) and those of the harmonic oscillator? The observed ones can be reproduced in an independent-particle model if to the harmonic-oscillator Hamiltonian $H_{\rm ho}$ a spin–orbit as well as an orbit–orbit term is added of the form

$$V_{\rm so} = \zeta_{\rm so}(r)\bar{\ell}\cdot\bar{s}, \qquad V_{\rm oo} = \zeta_{\rm oo}(r)\bar{\ell}\cdot\bar{\ell}. \tag{74}$$

The eigenvalue problem associated with the Hamiltonian $H_{\rm ho} + V_{\rm so} + V_{\rm oo}$ is not, in general, analytically solvable but the dominant characteristics can be found from the expectation values

$$\langle n\ell jm_j|V_{\rm so}|n\ell jm_j\rangle = \tfrac{1}{2}\langle\zeta_{\rm so}(r)\rangle_{n\ell}\left[j(j+1) - \ell(\ell+1) - \tfrac{3}{4}\right], \tag{75}$$

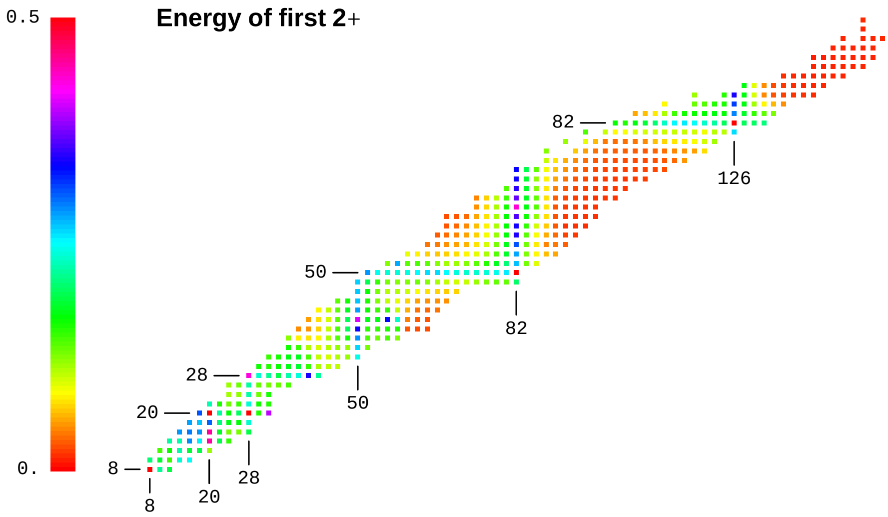

FIG. 7: The energy of the first-excited 2^+ state in all even–even nuclei with $N, Z \geq 8$ (where known experimentally) plotted as a function of neutron number N along the x axis and proton number Z along the y axis. The excitation energy is multiplied by $A^{1/3}$ and subsequently normalized to 1 for ^{208}Pb where this quantity is highest. The value of $E_\mathrm{x}(2_1^+)A^{1/3}$ is indicated by the scale shown on the left. To improve the resolution of the plot, the scale only covers part of the range from 0 to 0.5 since only a few doubly magic nuclei (^{16}O, ^{40,48}Ca, ^{132}Sn and ^{208}Pb) have values greater than 0.5.

and

$$\langle n\ell jm_j|V_\mathrm{oo}|n\ell jm_j\rangle = \ell(\ell+1)\langle\zeta_\mathrm{oo}(r)\rangle_{n\ell}, \tag{76}$$

with radial integrals defined as

$$\langle\zeta(r)\rangle_{n\ell} = \int_0^{+\infty} \zeta(r)R_{n\ell}(r)R_{n\ell}(r)r^2\,dr. \tag{77}$$

Consequently, the degeneracy of the single-particle levels within one major oscillator shell is lifted. Empirically, one finds that the radial integrals approximately satisfy the relations [16]

$$\langle\zeta_\mathrm{so}(r)\rangle_{n\ell} \approx -20A^{-2/3}\ \mathrm{MeV}, \qquad \langle\zeta_\mathrm{oo}(r)\rangle_{n\ell} \approx -0.1\ \mathrm{MeV}. \tag{78}$$

The origin of the orbit–orbit coupling can be understood from elementary arguments. The corrections to the harmonic-oscillator potential are repulsive for short and large distances and attractive for intermediate distances. These corrections therefore favor large-ℓ over small-ℓ orbits. The spin–orbit coupling has a relativistic origin. An important feature is that the radial integral is negative, reflecting the empirical finding that states with parallel spin and orbital angular momentum are pushed down in energy while in the antiparallel case they are pushed up.

The summary of the preceding discussion is that a simple approximation of the nuclear mean-field potential consists of a three-dimensional harmonic oscillator corrected with a spin–orbit and an orbit–orbit term. If, in addition, a two-body residual interaction is included, the many-body Hamiltonian that must be solved acquires the following form:

$$H = \sum_{k=1}^{A}\left(\frac{p_k^2}{2m_{\rm n}} + \frac{1}{2}m_{\rm n}\omega^2 r_k^2 + \zeta_{\rm oo}\,\bar{\ell}_k\cdot\bar{\ell}_k + \zeta_{\rm so}\,\bar{\ell}_k\cdot\bar{s}_k\right) + \sum_{k<l} V_{\rm res}(\xi_k,\xi_l), \tag{79}$$

where the indices in the second sum run over a *restricted* number of particles, usually only the valence nucleons. In spite of the severe simplifications of the original many-body problem (56), the solution of the Schrödinger equation associated with the Hamiltonian (79) still represents a formidable problem since the residual interaction must be diagonalized in a basis of Slater determinants of the type (63). Even if one limits oneself to valence-shell excitations, the dimension of the Hilbert space rapidly explodes with increasing mass of the nucleus. The m-scheme basis can be used to illustrate this. Because of the antisymmetry of Slater determinants, their number can be computed easily. For n neutrons and z protons distributed over Ω_n and Ω_z orbital states, respectively, the dimension of the basis is

$$\frac{\Omega_n!}{n!(\Omega_n - n)!}\frac{\Omega_z!}{z!(\Omega_z - z)!}. \tag{80}$$

Application of this formula to ^{28}Si (in the sd shell, $\Omega_n = \Omega_z = 12, n = z = 6$) and to ^{78}Y (half-way between the magic numbers 28 and 50, $\Omega_n = \Omega_z = 22, n = z = 11$) illustrates the point since it leads to dimensions of 8.5 10^5 and 5.0 10^{11}, respectively.

Given the considerable effort it takes to solve the nuclear many-body problem even only approximately, any analytical solution of (79) that can be obtained through symmetry techniques might be of considerable value. In fact, the residual interaction can approximately be written as pairing-plus-quadrupole,

$$V_{\rm res}(\xi_k,\xi_l) = V_{\rm pairing}(\bar{r}_k,\bar{r}_l) + V_{\rm quadrupole}(\bar{r}_k,\bar{r}_l), \tag{81}$$

where the exact form of these interactions is defined below. For particular values of the parameters in the mean field and if either the pairing or the quadrupole residual interaction is dominant, the eigenproblem (79) can be solved analytically. Three situations arise, of which two are of interest:

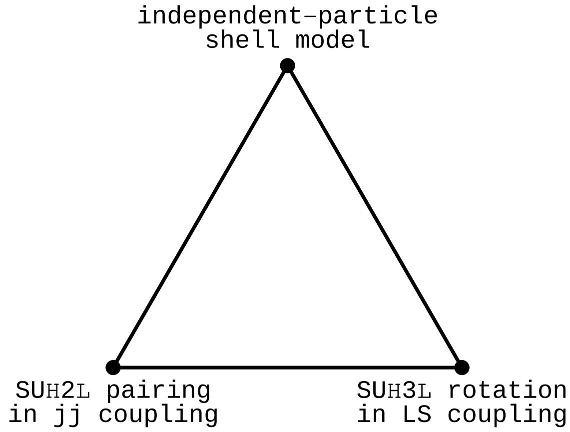

FIG. 8: Schematic representation of the shell-model parameter space with its three analytically solvable vertices.

1. *No residual interaction.* If $V_{\rm res}(\xi_k, \xi_l) = 0$, the solution of (79) reduces to a Slater determinant built from harmonic-oscillator eigenstates.

2. *Pairing interaction.* If the residual interaction has a pure pairing character, Racah's SU(2) model of pairing results. This model is usually applied in the jj-coupling limit of strong spin–orbit coupling.

3. *Quadrupole interaction.* If the residual interaction has a pure quadrupole character, Elliott's SU(3) model of rotation results. This model requires an LS-coupling scheme which occurs in the absence of spin–orbit coupling.

The situation is represented schematically in Fig. 8. It should be emphasized that, in contrast to the top vertex, the two bottom vertices, SU(2) and SU(3), represent solutions of the nuclear Hamiltonian which include genuine many-body correlations. These two limits are thus of particular interest. A brief summary of the pairing and quadrupole limits of the nuclear shell model is given in the following subsections. A more detailed review of the use of symmetries in the shell model has been given elsewhere [20].

B. Pairing correlations

The pairing interaction is a reasonable first-order approximation to the strong force between identical nucleons. For nucleons in a single-j shell the interaction is defined by the

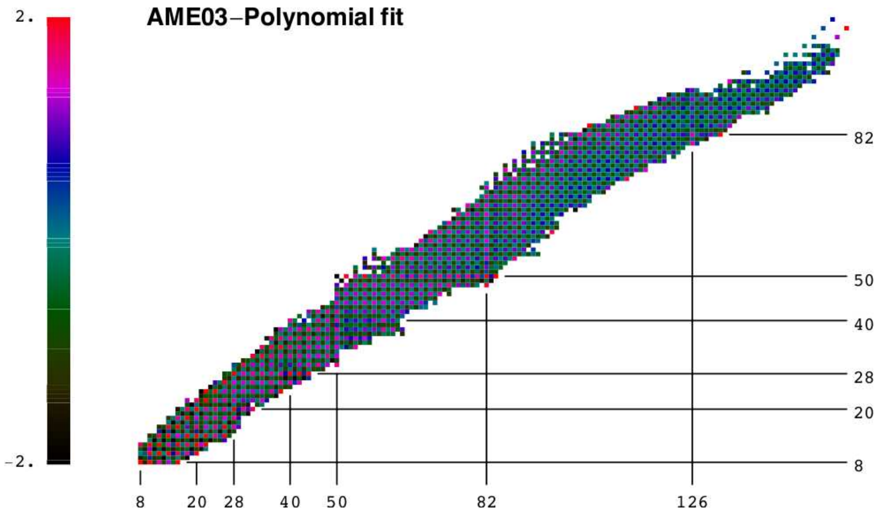

FIG. 9: The even–odd effect in nuclear binding energies as evidence for pairing correlations in nuclei. The difference between the experimental binding energies from the atomic-mass compilation of 2003 (AME03) [21] and a smooth local fit to these data is shown as a function of neutron number N along the x axis and proton number Z along the y axis. The local fit assumes a polynomial in N and Z, whose coefficients are determined from about 50 masses in the neighborhood.

matrix elements

$$\langle j^2; JM_J|V_{\rm pairing}|j^2; JM_J\rangle = -g_0(2j+1)\delta_{J0}, \tag{82}$$

where j is the total (orbital+spin) angular momentum of a single nucleon (hence j is half-odd-integer), J results from the coupling of two js and M_J is the projection of J on the z axis. Furthermore, g_0 is the strength of the interaction which is attractive in nuclei ($g > 0$).

Evidence for the pairing character of the interaction between identical nucleons can be obtained from simple arguments as is illustrated in Fig. 9. The figure shows the difference between the experimental nuclear binding energies and a smooth local fit to these data as a function of neutron and proton numbers N and Z. The local fit assumes a polynomial in N and Z, whose coefficients are determined to about 50 masses in the neighborhood. The details of this fit are unimportant for the present argument, except for the fact that no difference is made between even–even, odd-mass and odd–odd nuclei which are all fitted with the same polynomial. The figure clearly demonstrates the existence of an even–odd effect in the observed binding energies since even–even nuclei are systematically more bound than found in the polynomial fit while odd–odd nuclei are less bound. The simplest interpretation of this empirical finding is that there exists an attractive interaction between two identical nucleons.

The pairing interaction is less realistic than a short-range delta interaction but has the advantage that the corresponding many-body problem can be solved analytically. Furthermore, its analysis is important because it is at the basis of seniority [22] which has found fruitful application in nuclear physics with considerable empirical evidence in semi-magic nuclei.

The results can be summarized as follows. A state with n identical particles and diagonal in the pairing interaction, is characterized—in addition to the angular momentum J and its projection M_J—by a quantum number υ. The energy of this state is given by

$$E(n,\upsilon) = -\frac{1}{4}g_0(n-\upsilon)(2\Omega_j - n - \upsilon + 2), \tag{83}$$

where $2\Omega_j = 2j+1$. The quantum number υ *counts the number of particles not coupled to* $J=0$. Any state $|j^n \upsilon J M_J\rangle$ can be constructed from $|j^\upsilon \upsilon J M_J\rangle$ according to

$$|j^n \upsilon J M_J\rangle \propto \left(S_+^j\right)^{(n-\upsilon)/2} |j^\upsilon \upsilon J M_J\rangle, \tag{84}$$

where S_+^j is an operator which creates a pair of particles in the j shell with their angular momentum coupled to $J=0$. In other words, $|j^\upsilon \upsilon J M_J\rangle$ acts as a *parent* state for a whole class of states $|j^n \upsilon J M_J\rangle$ just by the action of the pair state S_+^j. For this reason, υ is called seniority.

The above results remain valid if the n identical particles are distributed over several *degenerate* j shells by making the substitutions $S_+^j \mapsto S_+ \equiv \sum_j S_+^j$ and $\Omega_j \mapsto \Omega = \sum_j \Omega_j$. In this form the pairing formalism can be used to make several characteristic predictions: a constant excitation energy (independent of n) of the first-excited 2^+ state in even–even isotopes, the linear variation of two-nucleon separation energies as a function of n, the odd–even staggering in nuclear binding energies, the enhancement of two-nucleon transfer.

The first of these predictions is illustrated in Fig. 10. The ground state of an even–even nucleus has $\upsilon = 0$ and the lowest excited states have $\upsilon = 2$. An example of such $\upsilon = 2$ states are those in a two-nucleon j^2 configuration with $J \neq 0$, $J = 2, 4, \ldots, 2j-1$. The energy difference between $\upsilon = 2$ and $\upsilon = 0$ states is given by

$$E(n,2) - E(n,0) = g_0\Omega, \tag{85}$$

and is independent of the number of valence nucleons. This prediction is illustrated in Fig. 10

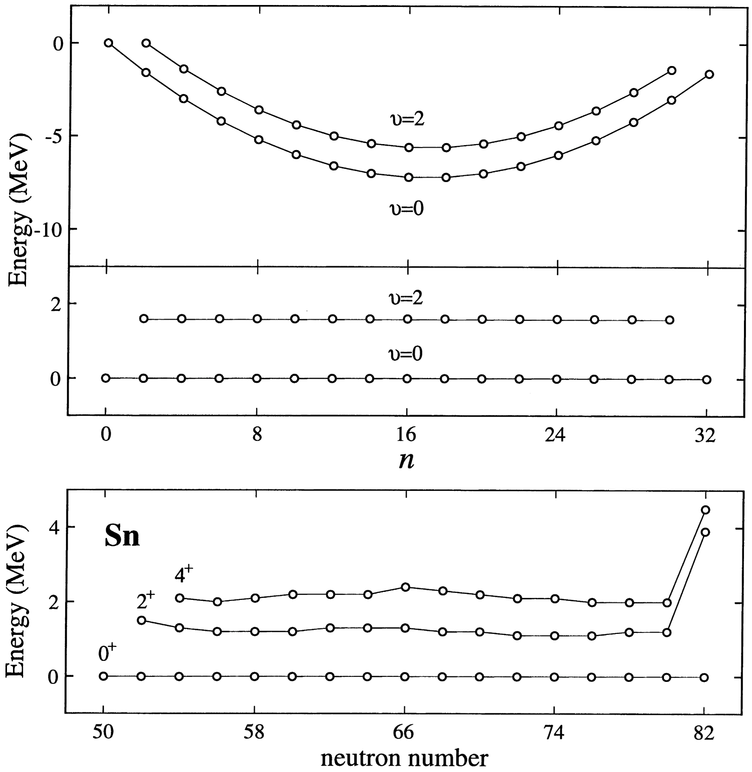

FIG. 10: The difference $E(n,2) - E(n,0)$ is a function of particle number n (top) and the corresponding observed excitation energies $E_{\mathrm{x}}(2_1^+) \equiv E(2_1^+) - E(0_1^+)$ and $E_{\mathrm{x}}(4_1^+) \equiv E(4_1^+) - E(0_1^+)$ in the Sn isotopes.

where it is compared with the excitation energies of the 2_1^+ and 4_1^+ levels in the even–even Sn isotopes.

The discussion of pairing correlations in nuclei traditionally has been inspired by the treatment of superfluidity in condensed matter [23, 24]. The superfluid phase in the latter systems is characterized by the presence of a large number of identical bosons in a single quantum state, which is called the condensate. In superconductors the bosons are pairs of electrons with opposite momenta that form at the Fermi surface. The character of the bosons in nuclei can be understood by analyzing the ground state of a pairing Hamiltonian. For an even–even nucleus, according to (84), it is given by

$$|j^n \upsilon = 0, J = M = 0\rangle \propto (S_+)^{n/2}\,|\mathrm{o}\rangle. \tag{86}$$

In nuclei the bosons are thus pairs of valence nucleons with opposite angular momenta.

C. Quadrupole correlations

The second class of analytically solvable shell-model Hamiltonians corresponds to the case of nucleons occupying an entire shell of the harmonic oscillator and interacting through a quadrupole force. In this case the Hamiltonian is of the form

$$H = \sum_{k=1}^{A} \left(\frac{p_k^2}{2m_{\mathrm{n}}} + \frac{1}{2} m_{\mathrm{n}} \omega^2 r_k^2 \right) - g_2 Q \cdot Q, \tag{87}$$

which contains a quadrupole operator

$$Q_\mu = \sqrt{\frac{3}{2}} \left[\sum_{k=1}^{A} \frac{1}{b^2} [\bar{r}_k \times \bar{r}_k]^{(2)}_\mu + \frac{b^2}{\hbar^2} \sum_{k=1}^{A} [\bar{p}_k \times \bar{p}_k]^{(2)}_\mu \right]. \tag{88}$$

Note that $Q \cdot Q \equiv \sum_\mu Q_\mu Q_\mu$ contains one-body ($k = l$) as well as two-body ($k \neq l$) terms.

The proof that the shell-model Hamiltonian (87) is analytically solvable was given by Elliott [25]. The reasons for its solvability are that the five components of the quadrupole operator (88) together with the three components of the angular momentum vector $\bar{L} = \sum_k (\bar{r}_k \wedge \bar{p}_k)$ form a closed algebra SU(3) and, furthermore, that these operators commute with the harmonic-oscillator Hamiltonian [*i.e.*, with the one-body term in (87)]. The quadrupole interaction is in fact a combination of Casimir operators,

$$Q \cdot Q = 4C_2[\mathrm{SU}(3)] - 3\bar{L}^2 = 4C_2[\mathrm{SU}(3)] - 3C_2[\mathrm{SO}(3)], \tag{89}$$

and it follows that the Hamiltonian (87) has the eigenvalues

$$E(\lambda, \mu, L) = E_0 - g_2 \left[4(\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu) - 3L(L+1) \right], \tag{90}$$

where E_0 is a constant energy associated with the first term in the Hamiltonian (87) and λ and μ label the SU(3) representations. The quadrupole interaction represents an example of symmetry breaking since the degeneracy associated with an entire oscillator shell is lifted by the quadrupole interaction.

The importance of Elliott's idea is that it gives rise to a rotational classification of states through mixing of spherical configurations. With the SU(3) model it was shown, for the first time, how deformed nuclear shapes may arise out of the spherical shell model. As a consequence, Elliott's work bridged the gap between the nuclear shell model and the liquid droplet model which up to that time (1958) existed as separate views of the nucleus.

VII. THE INTERACTING BOSON MODEL

Arguably more than any other model of the nucleus, the interacting boson model (IBM) illustrates the power of group-theoretical techniques and the physics insights that can be obtained from them. In this section a brief introduction to the IBM is given with the primary goal to provide an example of the notion of dynamical symmetry which was introduced in Sect. IV. It is not the aim here to give a full account of the IBM which can be found in the book of Iachello and Arima [26].

A. The model

The building blocks of the IBM are s and d bosons with angular momenta $\ell = 0$ and $\ell = 2$. A nucleus is characterized by a constant total number of bosons N which equals half the number of valence nucleons (particles or holes, whichever is smaller). In these lecture notes no distinction is made between neutron and proton bosons, an approximation which is known as IBM-1.

Since the Hamiltonian of the IBM-1 conserves the total number of bosons, it can be written in terms of the 36 operators $b^\dagger_{\ell m_\ell} b_{\ell' m'_\ell}$ where $b^\dagger_{\ell m_\ell}$ ($b_{\ell m_\ell}$) creates (annihilates) a boson with angular momentum ℓ and z projection m_ℓ. This set of 36 operators generates the Lie algebra U(6). A Hamiltonian that conserves the total number of bosons is of the generic form

$$H = E_0 + H_1 + H_2 + H_3 + \cdots, \tag{91}$$

where the index refers to the order of the interaction in the generators of U(6). The first term E_0 is a constant which represents the binding energy of the core. The second term is the one-body part

$$H_1 = \epsilon_s [s^\dagger \times \tilde{s}]^{(0)} + \epsilon_d \sqrt{5} [d^\dagger \times \tilde{d}]^{(0)} \equiv \epsilon_s n_s + \epsilon_d n_d, \tag{92}$$

where $\times$ refers to coupling in angular momentum, $\tilde{b}_{\ell m_\ell} \equiv (-)^{\ell - m_\ell} b_{\ell,-m_\ell}$ and the coefficients ϵ_s and ϵ_d are the energies of the s and d bosons. The third term in the Hamiltonian (91) represents the two-body interaction

$$H_2 = \sum_{\ell_1 \leq \ell_2, \ell'_1 \leq \ell'_2, L} \tilde{v}^L_{\ell_1 \ell_2 \ell'_1 \ell'_2} [[b^\dagger_{\ell_1} \times b^\dagger_{\ell_2}]^{(L)} \times [\tilde{b}_{\ell'_2} \times \tilde{b}_{\ell'_1}]^{(L)}]^{(0)}_0, \tag{93}$$

where the coefficients $\tilde{v}$ are related to the interaction matrix elements between normalized two-boson states,

$$\langle \ell_1 \ell_2; LM | H_2 | \ell'_1 \ell'_2; LM \rangle = \sqrt{\frac{(1+\delta_{\ell_1 \ell_2})(1+\delta_{\ell'_1 \ell'_2})}{2L+1}} \tilde{v}^L_{\ell_1 \ell_2 \ell'_1 \ell'_2}.$$

Since the bosons are necessarily symmetrically coupled, allowed two-boson states are s^2 ($L=0$), sd ($L=2$) and d^2 ($L=0,2,4$). Since for n states with a given angular momentum one has $n(n+1)/2$ interactions, seven independent two-body interactions v are found: three for $L=0$, three for $L=2$ and one for $L=4$.

This analysis can be extended to higher-order interactions. One may consider, for example, the three-body interactions $\langle \ell_1 \ell_2 \ell_3; LM | H_3 | \ell'_1 \ell'_2 \ell'_3; LM \rangle$. The allowed three-boson states are s^3 ($L=0$), $s^2 d$ ($L=2$), sd^2 ($L=0,2,4$) and d^3 ($L=0,2,3,4,6$), leading to $6+6+1+3+1=17$ independent three-body interactions for $L=0,2,3,4,6$, respectively.

B. Dynamical symmetries

The characteristics of the most general IBM Hamiltonian which includes up to two-body interactions and its group-theoretical properties are by now well understood [27]. Numerical procedures exist to obtain its eigensolutions but the problem can be solved analytically for particular choices of boson energies and boson–boson interactions. For an IBM Hamiltonian with up to two-body interactions between the bosons, three different analytical solutions or limits exist: the vibrational U(5) [28], the rotational SU(3) [29] and the γ-unstable SO(6) limit [30]. They are associated with the algebraic reductions

$$\mathrm{U}(6) \supset \left\{ \begin{array}{c} \mathrm{U}(5) \supset \mathrm{SO}(5) \\ \mathrm{SU}(3) \\ \mathrm{SO}(6) \supset \mathrm{SO}(5) \end{array} \right\} \supset \mathrm{SO}(3). \tag{94}$$

The algebras appearing in the lattice (94) are subalgebras of U(6) generated by operators of the type $b^\dagger_{\ell m_\ell} b_{\ell' m'_\ell}$, the explicit form of which is listed, for example, in Ref. [26]. With the subalgebras U(5), SU(3), SO(6), SO(5) and SO(3) there are associated one linear [of U(5)] and five quadratic Casimir operators. The total of all one- and two-body interactions can be represented by including in addition the operators $C_1[\mathrm{U}(6)]$, $C_2[\mathrm{U}(6)]$ and $C_1[\mathrm{U}(6)]C_1[\mathrm{U}(5)]$. The most general IBM Hamiltonian with up to two-body interactions can thus be written in an *exactly* equivalent way with Casimir operators. Specifically, the Hamiltonian reads

$$\begin{aligned} H_{1+2} &= \kappa_1 C_1[\mathrm{U}(5)] + \kappa_1' C_2[\mathrm{U}(5)] + \kappa_2 C_2[\mathrm{SU}(3)] \\ &\quad + \kappa_3 C_2[\mathrm{SO}(6)] + \kappa_4 C_2[\mathrm{SO}(5)] + \kappa_5 C_2[\mathrm{SO}(3)], \end{aligned} \tag{95}$$

which is just an alternative way of writing $H_1 + H_2$ of Eqs. (92,93) if interactions are omitted that contribute to the binding energy only.

The representation (95) is much more telling when it comes to the symmetry properties of the IBM Hamiltonian. If some of the coefficients κ_i vanish such that H_{1+2} contains Casimir operators of subalgebras belonging to a *single* reduction in the lattice (94), then the eigenvalue problem can be solved analytically. Three classes of spectrum generating Hamiltonians can thus be constructed of the form

$$\begin{aligned} \mathrm{U}(5) &: H_{1+2} = \kappa_1 C_1[\mathrm{U}(5)] + \kappa_1' C_2[\mathrm{U}(5)] + \kappa_4 C_2[\mathrm{SO}(5)] + \kappa_5 C_2[\mathrm{SO}(3)], \\ \mathrm{SU}(3) &: H_{1+2} = \kappa_2 C_2[\mathrm{SU}(3)] + \kappa_5 C_2[\mathrm{SO}(3)], \\ \mathrm{SO}(6) &: H_{1+2} = \kappa_3 C_2[\mathrm{SO}(6)] + \kappa_4 C_2[\mathrm{SO}(5)] + \kappa_5 C_2[\mathrm{SO}(3)]. \end{aligned} \tag{96}$$

In each of these limits the Hamiltonian is written as a sum of commuting operators and, as a consequence, the quantum numbers associated with the different Casimir operators are conserved. They can be summarized as follows:

$$\begin{array}{ccccccccc} \mathrm{U}(6) & \supset & \mathrm{U}(5) & \supset & \mathrm{SO}(5) & \supset & \mathrm{SO}(3) & \supset & \mathrm{SO}(2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [N] & & n_d & & \tau & & \nu_\Delta L & & M_L \end{array},$$

$$\begin{array}{ccccccc} \mathrm{U}(6) & \supset & \mathrm{SU}(3) & \supset & \mathrm{SO}(3) & \supset & \mathrm{SO}(2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [N] & & (\lambda,\mu) & & K_L L & & M_L \end{array},$$

$$\begin{array}{ccccccccc} \mathrm{U}(6) & \supset & \mathrm{SO}(6) & \supset & \mathrm{SO}(5) & \supset & \mathrm{SO}(3) & \supset & \mathrm{SO}(2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [N] & & \sigma & & \tau & & \nu_\Delta L & & M_L \end{array}. \tag{97}$$

Furthermore, for each of the three Hamiltonians in Eq. (96) an analytic eigenvalue expression is available,

$$\mathrm{U}(5) : E(n_d, \tau, L) = \kappa_1 n_d + \kappa_1' n_d(n_d + 4) + \kappa_4 \tau(\tau + 3) + \kappa_5 L(L + 1),$$

$$\begin{aligned}
\mathrm{SU}(3)\ &:\ E(\lambda,\mu,L)=\kappa_2(\lambda^2+\mu^2+\lambda\mu+3\lambda+3\mu)+\kappa_5 L(L+1),\\
\mathrm{SO}(6)\ &:\ E(\sigma,\tau,L)=\kappa_3\sigma(\sigma+4)+\kappa_4\tau(\tau+3)+\kappa_5 L(L+1).
\end{aligned}\tag{98}$$

One can add Casimir operators of U(6) to the Hamiltonians in Eq. (95) without breaking any of the symmetries. For a given nucleus they reduce to a constant contribution. They can be omitted if one is only interested in the spectrum of a single nucleus but they should be introduced if one calculates binding energies. Note that none of the Hamiltonians in Eq. (96) contains a Casimir operator of SO(2). This interaction breaks the SO(3) symmetry (lifts the M_L degeneracy) and would only be appropriate if the nucleus is placed in an external electric or magnetic field.

The dynamical symmetries of the IBM arise if combinations of certain coefficients κ_i in the Hamiltonian (95) vanish. The converse, however, cannot be said. Even if all parameters κ_i are non-zero, the Hamiltonian H_{1+2} still may exhibit a dynamical symmetry and be analytically solvable. This is a consequence of the existence of unitary transformations which preserve the eigenspectrum of the Hamiltonian H_{1+2} (and hence its analyticity properties) and which can be represented as transformations in the parameter space $\{\kappa_i\}$. A *systematic* procedure exists for finding such transformations or parameter symmetries [31] which can, in fact, be applied to any Hamiltonian describing a system of interacting bosons and/or fermions.

While a numerical solution of the shell-model eigenvalue problem in general rapidly becomes impossible with increasing particle number, the corresponding problem in the IBM with s and d bosons remains tractable at all times, requiring the diagonalization of matrices with dimension of the order of $\sim 10^2$. One of the main reasons for the success of the IBM is that it provides a workable, albeit approximate, scheme which allows a description of transitional nuclei with a few parameters.

C. Partial dynamical symmetries

As argued in Sect. II, a dynamical symmetry can be viewed as a generalization and refinement of the concept of symmetry. Its basic paradigm is to write a Hamiltonian in terms of Casimir operators of a set of nested algebras. Its hallmarks are (i) solvability of the complete spectrum, (ii) existence of exact quantum numbers for all eigenstates and (iii) pre-determined structure of the eigenfunctions, independent of the parameters in the

Hamiltonian. A further enlargement of these ideas is obtained by means of the concept of partial dynamical symmetry. The idea is to relax the conditions of *complete* solvability and this can be done in essentially two different ways:

1. *Some of the eigenstates keep all of the quantum numbers.* In this case the properties of solvability, good quantum numbers, and symmetry-dictated structure are fulfilled exactly, but only by a subset of eigenstates [32, 33].

2. *All eigenstates keep some of the quantum numbers.* In this case none of the eigenstates is solvable, yet some quantum numbers (of the conserved symmetries) are retained. In general, this type of partial dynamical symmetry arises if the Hamiltonian preserves some of the quantum numbers in a dynamical-symmetry classification while breaking others [34, 35].

Combinations of 1 and 2 are possible as well, for example, if some of the eigenstates keep some of the quantum numbers [36].

It should be emphasized that dynamical symmetry, be it partial or not, is a notion that is not restricted to a specific model but can be applied to any quantal system consisting of interacting particles. Quantum Hamiltonians with a partial dynamical symmetry can be constructed with general techniques and their existence is closely related to the order of the interaction among the particles. Applications of these concepts continue to be explored in all fields of physics.

D. Microscopy

The connection of the IBM with the shell model arises by identifying the s and d bosons with correlated (or Cooper) pairs formed by two nucleons in the valence shell coupled to angular momentum $J = 0$ and $J = 2$. There exists a rich and varied literature on general procedures to carry out boson mappings in which pairs of fermions are represented as bosons. They fall into two distinct classes. In the first one establishes a correspondence between boson and fermion operators by requiring them to have the same algebraic structure, that is, the same commutation relations. In the second class the correspondence is established rather between state vectors in both spaces. In each case further subclasses exist that differ in their technicalities (*e.g.*, the nature of the operator expansion or the hierarchy in the state correspondence). In the specific example at hand, namely the mapping between the IBM and the shell model, arguably the most successful procedure has been the so-called OAI mapping [37] which associates vectors based on a seniority [U(5)] hierarchy in fermion (boson) space. It has been used in highly complex situations that go well beyond the simple version of IBM-1 with just identical s and d bosons and which include, for example, neutron–proton $T = 1$ and $T = 0$ pairs [38, 39].

E. The classical limit

The connection of the IBM with the geometric model of the nucleus can be obtained on the basis of coherent-state formalism [40–42]. The central outcome of the formalism is that for any IBM-1 Hamiltonian a corresponding potential $V(\beta, \gamma)$ can be constructed where β and γ parametrize the intrinsic quadrupole deformation of the nucleus [43]. This procedure is known as the classical limit of the IBM.

The coherent states used for obtaining the classical limit of the IBM are of the form

$$|N; \alpha_\mu\rangle \propto \left(s^\dagger + \sum_\mu \alpha_\mu d^\dagger_\mu\right)^N |\mathrm{o}\rangle, \tag{99}$$

where $|\mathrm{o}\rangle$ is the boson vacuum and α_μ are five complex variables. These have the interpretation of (quadrupole) shape variables and their associated conjugate momenta. If one limits oneself to static problems, the α_μ can be taken as real; they specify a shape and are analogous to the shape variables of the droplet model of the nucleus [43]. The α_μ can be related to three Euler angles which define the orientation of an intrinsic frame of reference, and two intrinsic shape variables, β and γ, that parametrize quadrupole vibrations of the nuclear surface around an equilibrium shape. In terms of the latter variables, the coherent state (99) is rewritten as

$$|N; \beta\gamma\rangle \propto \left(s^\dagger + \beta\left[\cos\gamma\, d^\dagger_0 + \sqrt{\frac{1}{2}}\sin\gamma\,(d^\dagger_{-2} + d^\dagger_{+2})\right]\right)^N |\mathrm{o}\rangle. \tag{100}$$

The expectation value of the Hamiltonian (91) in this state can be determined by elementary methods [44] and yields a functional expression in β and γ which is identified with a potential $V(\beta, \gamma)$, familiar from the geometric model. The classical limit of the most general Hamiltonian (91) is found to be of the generic form

$$V(\beta, \gamma) = E_0 + \sum_{n\geq 1} \frac{N(N-1)\cdots(N-n+1)}{(1+\beta^2)^n} \sum_{kl} a^{(n)}_{kl}\beta^{2k+3l}\cos^l 3\gamma, \tag{101}$$

where the coefficients $a_{kl}^{(n)}$ can be expressed in terms of the single-boson energies and n-body interactions between the bosons.

A catastrophe analysis [45] of the potential surfaces in (β,γ) as a function of the Hamiltonian parameters determines the stability properties of these shapes. This analysis was carried out for the general IBM Hamiltonian with up to two-body interactions by López–Moreno and Castaños [46]. The results of this study are confirmed [47] if a simplified IBM Hamiltonian is considered of the form

$$H_{1+2} = \epsilon\, n_d + \kappa\, Q \cdot Q. \tag{102}$$

This Hamiltonian provides a simple parametrization of the essential features of nuclear structural evolution in terms of a vibrational term n_d (the number of d bosons) and a quadrupole interaction $Q \cdot Q$ with

$$Q_\mu = [s^\dagger \times \tilde{d} + d^\dagger \times \tilde{s}]^{(2)}_\mu + \chi [d^\dagger \times \tilde{d}]^{(2)}_\mu. \tag{103}$$

Besides an overall energy scale, the spectrum of the Hamiltonian (102) is determined by two parameters: the ratio ϵ/κ and χ. The three limits of the IBM are obtained with an appropriate choice of parameters: U(5) if $\kappa = 0$, $\mathrm{SU}_\pm(3)$ if $\epsilon = 0$ and $\chi = \pm\sqrt{7}/2$, and SO(6) if $\epsilon = 0$ and $\chi = 0$. One may thus represent the parameter space of the simplified IBM Hamiltonian (102) on a triangle with vertices that correspond to the three limits U(5), SU(3) and SO(6), and where arbitrary points correspond to specific values of ϵ/κ and χ. Since there are two possible choices for SU(3), $\chi = -\sqrt{7}/2$ and $\chi = +\sqrt{7}/2$, the triangle can be extended to cover both cases by allowing χ to take negative as well as positive values.

The geometric interpretation of any IBM Hamiltonian on the triangle can now be found from its expectation value in the coherent state (100) which for the particular Hamiltonian (102) gives

$$\begin{aligned} V(\beta,\gamma) \;=\; & \frac{N\epsilon\beta^2}{1+\beta^2} + \kappa \left[\frac{N(5+(1+\chi^2)\beta^2)}{1+\beta^2} \right. \\ & \left. + \frac{N(N-1)}{(1+\beta^2)^2} \left(\frac{2}{7}\chi^2\beta^4 - 4\sqrt{\frac{2}{7}}\chi\beta^3 \cos 3\gamma + 4\beta^2 \right) \right]. \end{aligned} \tag{104}$$

The catastrophe analysis of this surface is summarized with the phase diagram shown in Fig. 11. Analytically solvable limits are indicated by the dots. Two different SU(3) limits occur corresponding to two possible choices of the quadrupole operator, $\chi = \pm\sqrt{7}/2$. Close

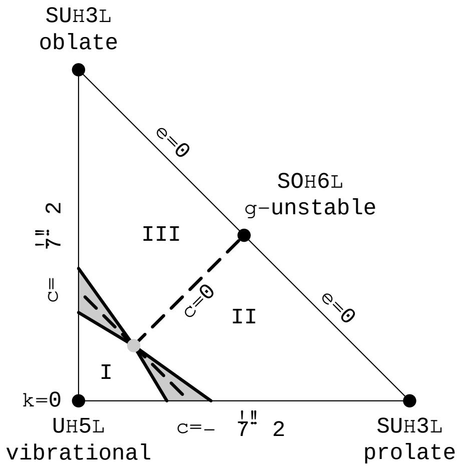

FIG. 11: Phase diagram of the Hamiltonian (102) and the associated geometric interpretation. The parameter space is divided into three regions depending on whether the corresponding potential has (I) a spherical, (II) a prolate deformed or (III) an oblate deformed absolute minimum. These regions are separated by dashed lines and meet in a triple point (grey dot). The shaded area corresponds to a region of coexistence of a spherical and a deformed minimum. Also indicated are the points on the triangle (black dots) which correspond to the dynamical-symmetry limits of the Hamiltonian (102) and the choice of parameters ϵ, κ and χ for specific points or lines of the diagram.

to the U(5) vertex, the IBM Hamiltonian has a vibrational-like spectrum. Towards the SU(3) and SO(6) vertices, it acquires rotational-like characteristics. This is confirmed by a study of the character of the potential surface in β and γ associated with each point of the triangle. In the region around U(5), corresponding to large ϵ/κ ratios, the minimum of the potential is at $\beta = 0$. On the other hand, close to the $SU_+(3)$–SO(6)–$SU_-(3)$ axis the IBM Hamiltonian corresponds to a potential with a deformed minimum. Furthermore, in the region around prolate $SU_-(3)$ ($\chi < 0$) the minimum occurs for $\gamma = 0^o$ while around oblate $SU_+(3)$ ($\chi > 0$) it does for $\gamma = 60^o$. In this way the picture emerges that the IBM parameter space can be divided into three regions according to the character of the associated potential having (I) a spherical minimum, (II) a prolate deformed minimum or (III) an oblate deformed minimum. The boundaries between the different regions (the so-called Maxwell set) are indicated by the dashed lines in Fig. 11 and meet in a triple point. The spherical–deformed border region displays another interesting phenomenon. Since the *absolute* minimum of the potential must be either spherical, or prolate or oblate deformed, its character uniquely determines the three regions and the dividing Maxwell lines. Nevertheless, this does not exclude the possibility that, in passing from one region to another, the potential may display a second *local* minimum. This indeed happens for the U(5)–SU(3) transition [48] where there is a narrow region of coexistence of a spherical and a deformed minimum, indicated by the shaded area in Fig. 11. Since, at the borders of this region of coexistence, the potential undergoes a *qualitative* change of character, the boundaries are genuine critical lines of the potential surface [45].

Although these geometric results have been obtained with reference to the simplified Hamiltonian (102) and its associated 'triangular' parameter space, they remain valid for the general IBM Hamiltonian with up to two-body interactions [46].

VIII. SUMMARY

In these lecture notes an introduction was given to the notions of symmetry and dynamical symmetry (or spectrum generating algebra). Their use in the solution of the (nuclear) many-body problem was described. Two particular examples of these techniques were discussed in detail: (i) SO(4) symmetry of the hydrogen atom and (ii) isospin symmetry in nuclei. A review was given of the shell model and the interacting boson model, with particular emphasis on the application of group-theoretical techniques in the context of these models.

Acknowledgment

This paper is dedicated to the memory of Marcos Moshinsky, the intellectual father of Mexican physics and founder of the *Escuela Latino Americana de Física*. The two years I have spent in Mexico as a visitor and the many hours with Marcos as a teacher, were crucial to my formation as a physicist. Without him I never could have given these lectures.

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[49] The notation with a tilde is used to distinguish the radial part of the wave function from the $R_{n\ell}(r)$ for the harmonic oscillator that will be encountered in Sect. VI.

[50] Throughout these lecture notes, small letters are normally reserved for operators associated with a single particle and capital letters for operators summed over many particles. This section deals with one-particle operators but for clarity's sake it is important to distinguish between operators, which shall be denoted in this section by capital letters L, P,. . . , and their associated labels, which shall be denoted by corresponding small letters ℓ, p,. . . .