Frequency-dependent reflection of a misaligned beam by a Fabry–Pérot cavity

Aline Cahuzac, Mathieu Gross, Eleonora Capocasa, Matteo Barsuglia, Raffaele Flaminio, Paolo La Penna

Frequency-dependent reflection of a misaligned beam by a Fabry-Perot cavity

Aline Cahuzac,[1, a] Mathieu Gross,[1, a] Eleonora Capocasa,[2] Matteo Barsuglia,[2] Raffaele Flaminio,[3] and Paolo La Penna[4]
[1)] *École Polytechnique, IP Paris, 91120 Palaiseau, France*
[2)] *Université Paris Cité, CNRS, Astroparticule et Cosmologie, F-75013 Paris, France*
[3)] *Laboratoire d'Annecy-le-Vieux de Physique des Particules (LAPP), Université Savoie Mont Blanc, CNRS/IN2P3, F-74941 Annecy-le-Vieux, France*
[4)] *ESO – European Southern Observatory, 85748 Garching bei München, Germany*

(Dated: February 2, 2024)

Gravitational-wave detectors as Virgo, LIGO and KAGRA are modified Michelson interferometers, with a system of coupled Fabry-Perot cavities, to increase its sensitivity and bandwidth. In order to control the detector, several radio-frequency sidebands, not resonant in the kilometric arms but resonant in the central part of the interferometer, are added to the carrier frequency to extract longitudinal and alignment error signals. Misalignment of the laser in the Fabry-Perot cavities causes sensitivity degradation through different mechanisms, and results in non-superposition of carrier and sidebands. These relative misalignment between fields at different frequency contain clues to optimally align the interferometer, but the question of the direction of a reflected beam by a Fabry-Perot cavity, as a function of the state of resonance of the incoming electromagnetic field, is neither straightforward nor intuitive. While numerical optical simulations used in the gravitational-wave detector community are able to answer the question, they do not give a qualitative and handy understanding of the observed phenomenon, useful for the commissioning and operation of the detectors. In this letter, we present a model based on first-order modal Gaussian beam development to calculate analytically how misalignment on the input beam in a Fabry-Perot cavity translates into misalignment of the reflected and circulating beams. We find a strong dependence on the beam resonance condition, but also on the mirror geometry. Finally, we checked the consistency of our model by comparing its predictions with existing numerical simulators.

In reflection of the coupled cavities constituting a gravitational-wave interferometer, we sometimes observe patterns where the laser field frequency components are no longer superposed in the transverse plane (they are on input). This effect is attributed to residual mismatch and misalignment. To understand this phenomenon in complex systems, the first step is to study for a single Fabry-Perot cavity.

Such a cavity of length L consists in two spherical mirrors M_i (input) and M_e (end) facing each other, with radii of curvatures R_i, R_e. The input beam is a monochromatic laser of wavelength λ_0. The cavity axis (Oz) goes through both centers of curvature (see figure 1).

Misalignment designates the differences (mainly shift and tilt) between the axes of the laser propagation, and of the cavity. For spherical mirrors, shift and tilt along (Ox) and (Oy) are independent at first order (small misalignment) and can be treated analogously (axial symmetry): we therefore reduce our 3-dimensional problem to a 2-dimension one in plane (x,z).

[a)] A. Cahuzac and M. Gross contributed equally to this work

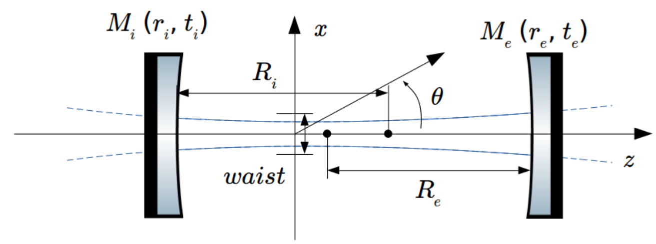

Figure 1. Fabry-Perot cavity and associated frame.

The caviy geometry (length L and radii of curvature R_i, R_e) determines the *beam shape* that can resonate in it (propagation axis, waist size w_0 and waist position z_0).

In paraxial approximation (moderate divergence) a laser field of wavelength λ_0 can be modeled as a Gaussian beam, characterized by a beam shape and a *transverse beam profile*.

A Hilbert basis of $\mathscr{E}$, the space of Gaussian beams with wavelength λ_0, is obtained by decomposing the transverse profile in Hermite-Gaussian functions with a chosen common beam shape. The basis vectors are Transverse Electromagnetic Modes (TEM$_{nm}$) with the form (conventionally $z_0 = 0$):

$$\psi^{\pm}_{nm}(x,y,z) = \psi^{\perp}_{nm}(x,y,z)e^{-i(\pm kz \mp (n+m+1)\phi(z))} \qquad (1)$$

with spatial pulsation k ($k_0 = 2\pi/\lambda_0$ for carrier frequency, $k = k_0 \pm 2\pi f/c$ for modulation sidebands). The sign is $+$ for forward beams (from the laser), $-$ for backward ones. $\psi^{\perp}_{nm}$ is slowly-varying in z and real for $z = 0$. The *Gouy phase shift*

$$\phi(z) = \arctan \frac{2z}{kw_0^2} \quad (2)$$

thus describes the phase difference between the fundamental Gaussian beam and the plane wave of same wavelength.

One beam shape makes the TEM$_{nm}$ eigenmodes of the cavity, *ie* eigenstates of the round-trip operator (seen as a linear operator $\mathscr{C} : \mathscr{E} \to \mathscr{E}$). Indeed, under the stability condition

$$0 < g = \left(1 - \frac{L}{R_i}\right)\left(1 - \frac{L}{R_e}\right) < 1$$

$\mathscr{C}$ is diagonalizable in the eigenbasis of Hermite-Gaussian TEM$_{nm}$ with axis (Oz) and a waist size and position depending only on R_i, R_e, L, λ_0.

$\mathscr{E}$ has two orthogonal components: forward-propagating beams $\mathscr{E}^+$ (basis ψ_{nm}^+), and backward-propagating ones $\mathscr{E}^-$ (basis ψ_{nm}^-). Bases $\psi_{nm}^\pm$ are related by the coordinate change $(x, z, \theta) \longrightarrow (\tilde{x}, \tilde{z}, \tilde{\theta}) = (x, -z, -\theta)$ (see figure 2):

$$\psi_{nm}^-(x, y, z) = \tilde{\psi}_{nm}^+(\tilde{x}, \tilde{y}, \tilde{z}) \quad (3)$$

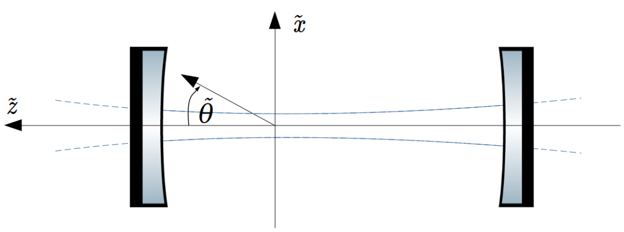

Figure 2. Backward propagation coordinates.

Physically, all modes have same electromagnetic frequency. Because of the accumulated Gouy phase shift (2), they resonate however for different cavity lengths and appear frequency-shifted on a cavity scan. Therefore, the cavity is said “resonant at λ_0" when TEM$_{00}$ is resonant, and a beam is “aligned and matched" when the input field is exactly the cavity's resonant TEM$_{00}$. From this configuration, we *misalign* the beam by tilting or shifting the propagation axis. Mismatch being a second-order modal perturbation[1], we here assume perfect matching (*ie* the laser waist is at $z = z_0 = 0$ and has radius w_0).

Beam misalignment is characterized by two numbers (see figure 3):

Tilt α: angle between cavity and beam axis

Shift a: distance between beam and cavity axes at $z = 0$ (waist)

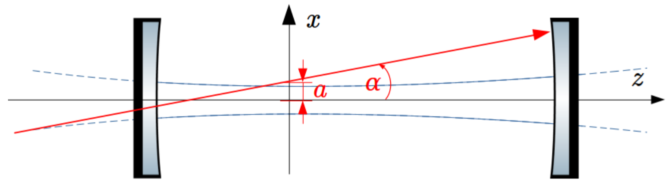

Figure 3. Misaligned beam.

A slightly misaligned beam can be decomposed using only modes 00 and 10 (first order):

$$E_{in} = \psi_{00}^+ + \left(\frac{a}{w_0} - i\frac{k_0 w_0}{2}\alpha\right)\psi_{10}^+ \quad (4)$$

(up to a global amplitude). The coupling coefficients are derived following the reasoning by Anderson[1] (their formula[1] contains a $+i$ instead of our $-i$, but $-i$ is coherent in our coordinate system and passes the test of Finesse and OSCAR comparison, unlike $+i$). This peculiar expression is of course only valid in a Hermite-Gauss basis.

A wave centered around $\mathbf{k} \neq k\hat{\mathbf{z}}$ is thus approximated by a combination of cavity eigenmodes axed on z. The beam axis position is encoded in the real and imaginary parts of the *relative amplitude* of both modes.

Usually, for a plane mirror M and plane waves, we define coefficients r, t between input, reflected and transmitted fields E_{in}, E_r, E_t as ratios between two complex fields in a fixed point (the mirror):

$$E_r\Big|_M = rE_{in}\Big|_M$$
$$E_t\Big|_M = itE_{in}\Big|_M$$

where r, t are real (we set the phase reference so that ψ^+ and ψ^- are in phase on M_i). Phase i is chosen to be coherent with the Finesse description (power continuity requires a phase of $\pm i$ between r and t).

The plane waves space $\mathscr{E}^\pm_{plane}$ being only 1-dimensional, linear operators are entirely defined by these proportionality coefficients. With the infinite-dimensional $\mathscr{E}$, these operators are *a priori* not proportional to the identity (or a “pseudo-identity" between bases $\psi_{nm}^\pm$) and we need to write their action on each mode TEM$_{nm}$. Since different TEMs are only in phase at the waist, we consider

$$\left.\frac{E^{out}_{n'm'}}{E^{in}_{nm}}\right|_{waist} \quad (5)$$

where E_{nm} is the single-mode component of index n, m in the field E. In the following, M_i, M_e are not necessarily plane.

Consider first transmission $\mathscr{T}_a$ through M_a ($a \in \{i,e\}$ placed at z_a. For zero-thickness mirrors, it should conserve the wavefront and thus be proportional to the identity:

$$\mathscr{T}_a : \psi^{\pm}_{nm} \mapsto it_a \psi^{\pm}_{nm}$$

with transmittivity coefficients $t_i, t_e \in \mathbb{R}$. In a high-finesse cavity, $t_e, t_i \ll 1$. We also define reflectivity coefficients $r_i, r_e \in \mathbb{R}$ such that $r_a^2 + t_a^2 = 1$ (power conservation) and:

$$E_r \Big|_{M_a} = r_a E_{in} \Big|_{M_a} \tag{6}$$

Indeed, *local* behaviour matches plane reflection-transmission, since we can always adopt a local plane approximation.

Local considerations do not immediately give global modal descriptions for transmission and reflection; they are only boundary condition.

Reflection operator $\mathscr{R}_a$ should be proportional to r_a, but not to the identity since it relates disconnected subspaces $\mathscr{E}^{\pm}$ and $\mathscr{E}^{-}$. We can however decompose it into two operators $\mathscr{R}_a^+ : \mathscr{E}^+ \to \mathscr{E}^-$ and $\mathscr{R}_a^- : \mathscr{E}^- \to \mathscr{E}^+$ and express them in the bases $\psi^{\pm}_{nm}$.

The intuitive idea (justified by full calculations in ABCD formalism[2]) is that, for an incident wave $\psi^{\pm}$, (6) determines the reflected field on a transverse surface (the mirror), which is enough to derive its decomposition in the basis $\psi^{\mp}_{nm}$.

More specifically, conservation of the transverse beam profile implies that $\mathscr{R}_a^+$ is *diagonal* in the bases $(\psi^+_{nm}, \psi^-_{nm})$. For such a diagonal operator, we call "pseudo-eigenvalues" coefficients $r_a^+(\psi^+_{nm})$ associated to "pseudo-eigenmode" pairs $(\psi^+_{nm}, \psi^-_{nm})$:

$$\mathscr{R}_a^+ \psi^+_{nm} = r_a^+(\psi^+_{nm}) \psi^-_{nm}$$

Boundary conditions then require phase equality of the incident and reflected beams at the mirror, but $\psi^{\pm}_{nm}$ has, here, a phase $e^{\mp i(kz_a - (n+m+1)\phi(z_a))}$. To restore phase continuity, necessarily

$$r_a(\psi^+_{nm}) = r_a e^{-i2(kz_a - (n+m+1)\phi(z_a))}$$

In particular, the reflection operator is therefore *not proportional to the pseudo-identity*.

Reflection from the other side $\mathscr{R}_a^-$ has pseudo-eigenvalues

$$r_a(\psi^-_{nm}) = r_a e^{-i2(kz_a + (n+m+1)\phi(z_a))}$$

Note that the propagation phase $2kz_a$ does not change sign.

Using this description for individual mirrors, we can now obtain relations between various fields in and around a Fabry-Perot cavity. In a plane cavity, we compute them considering all possible reflection-transmission combinations, introducing the *half-round-trip phase* $\varphi_{cav} = kL$:

- Reflection coefficient:

$$r_{cav} = \frac{E_r}{E_{in}} \Big|_{M_i} = r_i - \frac{t_i^2 r_e e^{-i2\varphi_{cav}}}{1 - r_i r_e e^{-i2\varphi_{cav}}} \tag{7}$$

- Transmission coefficient:

$$t_{cav} = \frac{E_t}{E_{in}} \Big|_{M_e} = -\frac{t_i t_e e^{-i\varphi_{cav}}}{1 - r_i r_e e^{-i2\varphi_{cav}}} \tag{8}$$

- Forward circulating coefficient (along $+z$)

$$c^+_{cav} = \frac{E_c^+}{E_{in}} \Big|_{M_i} = \frac{it_i}{1 - r_i r_e e^{-i2\varphi_{cav}}} \tag{9}$$

- Backward circulating coefficient (along $-z$)

$$c^-_{cav} = \frac{E_c^-}{E_{in}} \Big|_{M_i} = \frac{it_i r_e e^{-i\varphi_{cav}}}{1 - r_i r_e e^{-i2\varphi_{cav}}} \tag{10}$$

They depend *only on* φ_{cav}. It has been suggested[3] to adapt (7) to (10) to curved-mirror cavities using the φ_{cav}*-adaptation*

$$\varphi_{cav}^{plane} = kL \longrightarrow \varphi_{cav}^{nm} = kL - (n+m+1)\phi_G$$

where the additional phase $(n+m+1)\phi_G$ is due to Gouy phase shift (2):

$$\phi_G = \phi(z_e) - \phi(z_i) = \arctan \frac{2z_e}{kw_0^2} - \arctan \frac{2z_i}{kw_0^2} \tag{11}$$

However, a problem arises at $r_i \longrightarrow 1$: all modes being reflected with same coefficient 1, the outgoing beam has in basis ψ^-_{nm} exact same expression as the incoming beam in basis ψ^+_{nm}. This describes a plane reflection at the waist, which contradicts geometric intuition. Besides, while ϕ_G in the round-trip phase ensures beam phase continuity over one round-trip, on single reflection ($r_i = 1$) we are again confronted with a mode-dependent discontinuity (for a phase discontinuity to be physically acceptable, it should be intrinsic to the mirror and independent of the mode number).

As a satisfying upshot of this description, mirrors now introduce naturally the round-trip phase: instead of describing a double reflection as

1. propagation $z_i \longrightarrow z_e$
2. reflection with coefficient r_e
3. propagation $z_i \longleftarrow z_e$
4. reflection with coefficient r_i

we now have

1. reflection with r_e at z_e
2. reflection with r_i at z_i

We keep the notation φ_{cav} because the round-trip phase is the tuning variable in cavity frequency scans.

We can now compute *cavity operators* in the mirror operators algebra. For instance, the cavity operator $\mathscr{C}$ is $\mathscr{C} = \mathscr{R}_i\mathscr{R}_e$ of eigenvalues $r_i r_e e^{-i2(\phi(z_e)-\phi(z_i))}$. This algebraic method gives, in fact, the same results as substituting r,t in plane-cavity expressions with corresponding pseudo-eigenvalues.

We obtain the expressions of four diagonal operators in our chosen bases. To recover (7) for TEM$_{00}$ and simpler formulas, we redefine our ψ^-_{nm} basis by absorbing in each mode a common phase $e^{-i2(kz_i-\phi(z_i))}$, so that (3) becomes

$$\psi^-_{nm}(x,y,z) = e^{-i2(kz_i-\phi(z_i))}\,\tilde{\psi}^+_{nm}(\tilde{x},\tilde{y},\tilde{z}) \qquad (12)$$

(keeping notation ψ^-_{nm} after redefinition to avoid unnecessary heaviness). Nothing changes in the previous reasoning: the Hermite-Gauss property is preserved, and only *relative* amplitude and phase within a beam modal decomposition (involving one direction only) matter.

Finally, the cavity operators in our coupled bases $\psi^\pm_{nm}$ have following (pseudo-)eigenvalues:

- Reflection $\mathscr{R}$:

$$r_{cav}(\psi^+_{nm}) = e^{i2(n+m)\phi(z_i)}\left(r_i - \frac{t_i^2 r_e e^{-i2\varphi_{cav}}}{1-r_i r_e e^{-i2\varphi_{cav}}}\right) \qquad (13)$$

- Transmission $\mathscr{T}$:

$$t_{cav}(\psi^+_{nm}) = -\frac{t_i t_e e^{-i\varphi_{cav}}}{1-r_i r_e e^{-i2\varphi_{cav}}} \qquad (14)$$

- Forward circulating field $\mathscr{C}^+_{cav}$:

$$c^+_{cav}(\psi^+_{nm}) = \frac{it_i}{1-r_i r_e e^{-i2\varphi_{cav}}} \qquad (15)$$

- Backward circulating field $\mathscr{C}^-_{cav}$:

$$c^-_{cav}(\psi^+_{nm}) = e^{i2(n+m)\phi(z_i)}\frac{it_i r_e e^{-i2\varphi_{cav}}}{1-r_i r_e e^{-i2\varphi_{cav}}} \qquad (16)$$

t_{cav}, c^+_{cav} keep their plane-wave expression (*ie* only depend on φ_{cav}) but r_{cav}, c^-_{cav} get an additional phase $e^{2i(n+m)\phi(z_i)}$: while a plane-mirror cavity operates a faithful basis change $\psi^+_{nm} \mapsto \psi^-_{nm}$, a curved-mirror cavity sends basis ψ^+_{nm} to the "twisted" basis $e^{2i(n+m)\phi(z_i))}\psi^-_{nm}$.

In the particular case of a plano-concave cavity with flat M_i, $\phi(z_i) = 0$ (waist on the flat mirror) and everything only depends on φ_{cav}. This is in fact the condition for φ_{cav}-adaptation to work (VIRGO once had plane input mirrors).

As consistency check, consider reflection on a single curved mirror.

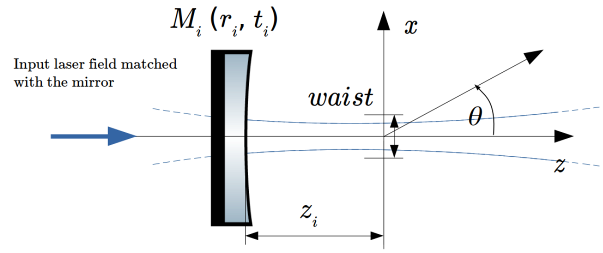

Figure 4. Chosen beam shape and frame.

Although there is no cavity here, a laser field can be said "matched to the mirror M" when M is an isophase surface of its TEM$_{00}$ (see figure 4): many beams can match the mirror but this arbitrary choice has no influence on final results.

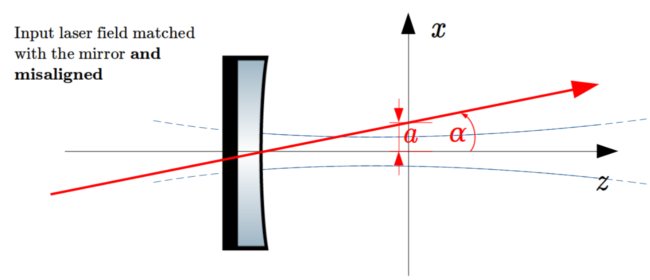

Figure 5. Misaligned beam on curved mirror.

For better intuition, we choose a matched beam basis and corresponding frame in which tilt a and shift α satisfy $a = -z_i\alpha$ (always possible for a spherically symmetric mirror): the beam thus hits M on the z-axis with angle α (see figure 5). At first order in α as in (4),

$$E_{in} = \psi^+_{00} + \left(-\frac{z_i\alpha}{w_0} - i\frac{k_0 w_0}{2}\alpha\right)\psi^+_{10} \qquad (17)$$

Applying reflection on each mode ($r_i = 1$),

$$\psi^+_{00} \mapsto \psi^-_{00} \qquad\qquad \psi^+_{10} \mapsto e^{2i\phi(z_i)}\psi^-_{10}$$

we get:

$$\begin{aligned}
E_r &= \psi^-_{00} + e^{2i\phi(z_i)}\left(\frac{a}{w_0} - i\frac{k_0 w_0}{2}\alpha\right)\psi^-_{10} \\
&= \psi^-_{00} + \left(\frac{1}{w_0}\left(a\cos 2\phi(z_i) + \frac{k_0 w_0^2\alpha}{2}\sin 2\phi(z_i)\right)\right. \\
&\quad \left. - i\frac{k_0 w_0}{2}\left(-\frac{2a}{k_0 w_0^2}\sin 2\phi(z_i) + \alpha\cos 2\phi(z_i)\right)\right)\psi^-_{10}
\end{aligned}$$

Note that mode amplitudes do not tell us anything as yet about the reflected beam direction, since we only derived coupling coefficients (4) for the *forward* basis ψ^+_{nm}. A simple way

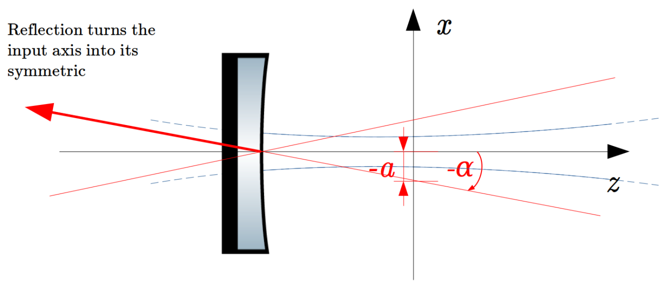

Figure 6. Reflected by single mirror.

to deduce the reflected beam axis is then to change frame from (x,z,θ) to $(\tilde{x},\tilde{z},\tilde{\theta})$ and express the field in basis $\tilde{\psi}^+_{nm}$ (forward basis if the laser came "from the right"). ψ^-_{nm} and $\tilde{\psi}^+_{nm}$ describe the same field, and since mode amplitudes are dimensionless numbers that cannot depend on the coordinates (a,α were measured in the same frame as w_0,k_0), E_r becomes in these new coordinates:

$$\tilde{E}_r = \tilde{\psi}^+_{00} + \left(\frac{1}{w_0}\left(a\cos 2\phi(z_i) + \frac{k_0 w_0^2\alpha}{2}\sin 2\phi(z_i)\right) - i\frac{k_0 w_0}{2}\left(-\frac{2a}{k_0 w_0^2}\sin 2\phi(z_i) + \alpha\cos 2\phi(z_i)\right)\right)\tilde{\psi}^+_{10}$$

Now, $\tilde{\psi}^+_{nm}$ is a forward Hermite-Gauss basis in coordinates $(\tilde{x},\tilde{z},\tilde{\theta})$. Therefore, modal decomposition coefficients are related to misalignment parameters by (4). We deduce the reflected beam shift $\delta\tilde{x} = a\cos 2\phi(z_i) + \frac{k_0 w_0^2\alpha}{2}\sin 2\phi(z_i)$ and tilt $\delta\tilde{\theta} = -\frac{2a}{k_0 w_0^2}\sin 2\phi(z_i) + \alpha\cos 2\phi(z_i)$. In (x,θ,z) coordinates, this is a shift $\delta x = \delta\tilde{x}$ and tilt $\delta\theta = -\delta\tilde{\theta}$

Using some trigonometry, the Gouy phase shift (2) and $a = -z_i\alpha$:

$$\delta x = a\cos 2\phi(z_i) + \frac{k_0 w_0^2\alpha}{2}\sin 2\phi(z_i) = z_i\alpha = -a$$

$$\delta\theta = -\alpha$$

The reflected axis turns out to be symmetric to the input one with respect to Oz (see figure 6), which we expected from Snell-Descartes' law (Oz being the normal to M at the entry point) but was not reproduced with φ_{cav}-adaptation.

The reflected direction is independent of w_0 or z_i (beam shape) and agrees with the limit of plane isophases ($w_0 \longrightarrow +\infty$, $z_i \longrightarrow 0$).

Consider now a misaligned beam sent in a cavity. Coefficients (13) to (16) should then be computed with the values characterizing this cavity. However, for a qualitative discussion we can write the coefficients in two specific cases: resonant and anti-resonant modes. Using (13) to (16), higher-order mode coefficients are easily deduced from those of TEM_{00} computed below.

(13) yields upon resonance ($\varphi_{cav} \equiv 0[\pi]$)

$$r_{cav} = r_i - \frac{t_i^2 r_e}{1 - r_i r_e}$$

With $r_e \sim 1$ and $r_a^2 + t_a^2 \sim 1$ (lossless mirrors),

$$r_{cav} = -1$$

Similarly using $t_i = \sqrt{1 - r_i^2}$ in (15) and (16),

$$\begin{aligned} c^+_{cav} &= \frac{i t_i}{1 - r_i r_e} \underset{r_e\to 1}{\longrightarrow} i\sqrt{\frac{1+r_i}{1-r_i}} \\ c^-_{cav} &= r_e c^+_{cav} \underset{r_e\to 1}{\longrightarrow} i\sqrt{\frac{1+r_i}{1-r_i}} \end{aligned} \quad (18)$$

On anti-resonance ($e^{-i2\varphi_{cav}} = -1$), (13) reduces to

$$r_{cav} = r_i + \frac{t_i^2 r_e}{1 + r_i r_e} \underset{r_e\to 1}{\longrightarrow} 1$$

As to the circulating field,

$$\begin{aligned} c^+_{cav} &= \frac{i t_i}{1 + r_i r_e} \underset{r_e\to 1}{\longrightarrow} i\sqrt{\frac{1-r_i}{1+r_i}} \\ c^-_{cav} &= -r_e c^+_{cav} \underset{r_e\to 1}{\longrightarrow} -i\sqrt{\frac{1-r_i}{1+r_i}} \end{aligned} \quad (19)$$

The narrow bandwidth (high-finesse assumption) suggests to consider all *non-resonant* frequencies as *anti-resonant*. When TEM_{00} resonates, higher order modes are out of resonance ($\phi(z)$-induced phase shift). Thus, we will discuss two cases:

i *Non-resonant beam*: both modes anti-resonant.

ii *Resonant beam*: resonant TEM_{00}, anti-resonant TEM_{10}.

Hopefully, most resonance conditions can be approximated by (i) or (ii): take as example figure 7.

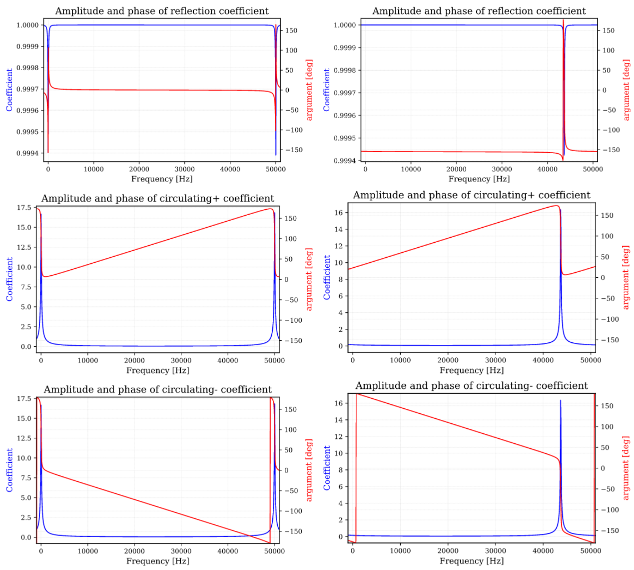

Figure 7. Variation of amplitude (blue) and phase (red) of reflection (top), forward-circulating (middle) and backward-circulating (bottom) coefficients for TEM_{00} (left) and TEM_{10} (right), in a model of VIRGO arm (finesse ~ 450).

r_{cav} goes indeed quickly to its anti-resonance value when leaving the exact resonance. Our two cases cover most of the spectrum.

Circulating coefficients, contrarily, exhibit a non-negligible phase slope between two consecutive resonances. Information regarding the phase should be treated with caution. Resonant beams behave as (ii) for a Gouy phase $2\phi_G \simeq \pi/2$ (long cavity or very curved mirrors), whereas non-resonant beams behave as (i) when on the contrary $2\phi_G \longrightarrow 0$ (plane or very short cavity). In VIRGO arms, $2\phi_G \simeq \pi/4$, recalling that

$$\phi_G = \arctan \frac{2z_e}{kw_0^2} - \arctan \frac{2z_i}{kw_0^2}$$

We now investigate the case of reflection on a cavity (basis and frame now fixed without ambiguity).

In case (i) out of resonance, neither fundamental nor higher-order modes resonate. We apply coefficients (19) to decomposition (4) and obtain the reflected beam tilt and shift (20):

$$\begin{aligned} \delta x &= a\cos 2\phi(z_i) + \frac{k_0 w_0^2 \alpha}{2} \sin 2\phi(z_i) \\ \delta\theta &= \frac{2a}{k_0 w_0^2} \sin 2\phi(z_i) - \alpha \cos 2\phi(z_i) \end{aligned} \qquad (20)$$

In general, no relation between a and α can be assumed in order to simplify these expressions. Nevertheless, the displacement on the input mirror verifies:

$$\delta x + z_i \delta\theta = a + z_i \alpha$$

Thus input and exit point of the anti-resonant beam on M_i coincide. This is rather expected since the coefficients applied are identical to a reflection by M_i (without cavity): the anti-resonant beam is geometrically reflected by M_i.

In case (ii), coefficients (18) prescribe phase -1 for TEM_{00} (now resonant), *ie* a phase change of π compared to case (i), while TEM_{10} is still anti-resonant. Therefore, with respect to case (i), the relative amplitude between both modes changes sign and, consequently, so do tilt and shift (21):

$$\begin{aligned} \delta x &= -a\cos 2\phi(z_i) - \frac{k_0 w_0^2 \alpha}{2} \sin 2\phi(z_i) \\ \delta\theta &= -\frac{2a}{k_0 w_0^2} \sin 2\phi(z_i) + \alpha \cos 2\phi(z_i) \end{aligned} \qquad (21)$$

Computing $\delta x + z_i \delta\theta$ shows that the resonant reflected beam's exit point on M_i is symmetric to its entry point.

The same method applies to the field circulating within the cavity, using (18) and (19). In the resonant beam case, the amplitude of TEM_{10} is suppressed by $(1+r_i)/(1-r_i)$ relative to TEM_{00}, so that both forward and backward beams practically coincide with the cavity axis (since $1 - r_i \ll 1$). In the non-resonant one, the relative amplitude (not phase) is conserved, which gives a misalignment of the same order of magnitude as the input one's, although possibly quite a different mixing of tilt and shift.

We notice that the anti-resonant, forward circulating beam follows exactly the input beam axis, since $\mathscr{C}^+_{cav}$ depend only on φ_{cav}. However, this is only valid at small ϕ_G. We will not draw conclusions from phase factors (giving the mix of tilt and shift) on circulating fields, since we know them to be hardly exploitable in our binary resonance/anti-resonance approximation.

We expect our order-1 approach to agree at small misalignment with higher-order simulations. Indeed, comparisons yielded a perfect match with Finesse2[4], a software also based on Hermite-Gauss modal decomposition[5], but also with the other simulation tool OSCAR[6] working on a different principle, confirming that our results are not an artefact of Gaussian beam approaches.

To summarize, we derived a first-order expression of shift and tilt for a misaligned beam reflected by a Fabry-Perot cavity, conditioned by the input beam resonance state. This was achieved by adapting the equivalent of reflection and transmission coefficients, in plano-concave cavities, to a general curved-mirror configuration. We applied this model to various examples and found for instance that a non-resonant beam is geometrically reflected by the input mirror, whereas a resonant input beam and its reflection have on the input mirror distinct beam center, mutually symmetric with respect to the cavity axis. Our first-order development approach may be complementary to numerical simulations to interpret, in particular, the relative misalignement between sidebands (non resonant) and carrier (resonant) in gravitational-wave detectors, but also in any optical setup using resonant Fabry-Perot cavities.

AKNOWLEDGEMENT

We would like to thank the colleagues of LIGO, Virgo, KAGRA collaboration for useful discussion and experts of Finesse and OSCAR for the support in the simulations. This work has been supported by LabEx UnivEarthS (ANR-10-LABX-0023 and ANR-18-IDEX-0001). AC and MG would like to thank European gravitational-wave observatory for financial support during their internship.

REFERENCES

[1] D. Anderson, "Alignment of resonant optical cavities," Applied Optics **23(17)**, 2944 (1984).

[2] H. Kogelnik, "Laser beams and resonators," Applied Optics **5(10)**, 1550–67 (1966).

[3] F. Calloni, M. Laval, J. Marque, and P. Ruggi, "Wavefront Signal Extraction and TCS," VIRGO (European Gravitational Observatory) Internal report on the reflection of a Gaussian wavefront by a cavity (implicitly assumed to be plano-concave).

[4] A. Freise, *Finesse2 handbook* (2014).

[5] F. Bayer-Helms, "Coupling coefficients of an incident wave and the modes of a spherical optical resonator in the case of mismatching an misalignment," Applied Optics **23(9)**, 1369–80 (1984).

[6] J. Degallaix, "OSCAR – A Matlab-based optical FFT code," (IOP, 2010).