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# Pionium lifetime and $\pi\pi$ scattering lengths in generalized chiral perturbation theory

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The relationship between the pionium lifetime and the  $\pi\pi$  scattering lengths is established, including the sizable electromagnetic corrections. The bound state formalism that is used is that of constraint theory which provides a covariant three-dimensional reduction of the Bethe–Salpeter equation. The framework of generalized chiral perturbation theory allows then an analysis of the lifetime value as a function of the  $\pi\pi$  scattering lengths, the latter being dependent on the quark condensate value.

The possible measurement of the pionium ( $\pi^+\pi^-$  atom) lifetime with a 10% precision in the DIRAC experiment at CERN [1] is expected to allow a determination of the combination  $(a_0^0 - a_0^2)$  of the  $\pi\pi$  scattering lengths with 5% accuracy; here,  $a_0^I$  is the strong interaction (dimensionless)  $S$ -wave scattering length in the isospin  $I$  channel. The strong interaction scattering lengths  $a_0^0$  and  $a_0^2$  have been evaluated in the literature in the framework of chiral perturbation theory ( $\chi PT$ ) to two-loop order of the chiral effective lagrangian [2–4]. Therefore, the pionium lifetime measurement provides a high precision experimental test of chiral perturbation theory predictions.

The nonrelativistic formula of the pionium lifetime in lowest order of electromagnetic interactions was first evaluated by Deser *et al.* [5]. It reads:

$$\frac{1}{\tau_0} = \Gamma_0 = \frac{16\pi}{9} \sqrt{\frac{2\Delta m_\pi}{m_{\pi^+}} \frac{(a_0^0 - a_0^2)^2}{m_{\pi^+}^2}} |\psi_{+-}(0)|^2, \quad (1)$$

where  $\Delta m_\pi = m_{\pi^+} - m_{\pi^0}$  and  $\psi_{+-}(0)$  is the wave function of the pionium at the origin (in  $x$ -space).

A precise comparison of the theoretical values of the strong interaction scattering lengths with experimental data necessitates, however, an evaluation of the corrections of order  $O(\alpha)$  ( $\alpha$  being the fine structure constant) to the above formula. Such an evaluation was recently done by several authors. In the frameworks of quantum field the-

ory and  $\chi PT$ , three different methods of evaluation have led to the same estimate, of the order of 6%, of these corrections [6–9]. The first method uses a three-dimensionally reduced form of the Bethe–Salpeter equation (constraint theory approach) and deals with an off-mass shell formalism [6,7]. The second method uses the Bethe–Salpeter equation with the Coulomb gauge [8]. The third one uses the approach of nonrelativistic effective theory [9].

The pionium lifetime, with the sizable  $O(\alpha)$  corrections included in, can be represented as:

$$\begin{aligned} \frac{1}{\tau} = \Gamma &= \frac{1}{64\pi m_{\pi^+}^2} (\mathcal{R}e\widetilde{\mathcal{M}}_{00,+})^2 (1 + \gamma) \\ &\times |\psi_{+-}(0)|^2 \sqrt{\frac{2\Delta m_\pi}{m_{\pi^+}} \left(1 - \frac{\Delta m_\pi}{2m_{\pi^+}}\right)} \\ &\equiv \Gamma_0 \sqrt{\left(1 - \frac{\Delta m_\pi}{2m_{\pi^+}}\right)} \left(1 + \frac{\Delta\Gamma}{\Gamma_0}\right), \quad (2) \end{aligned}$$

where  $\mathcal{R}e\widetilde{\mathcal{M}}_{00,+}$  is the real part of the on-mass shell scattering amplitude of the process  $\pi^+\pi^- \rightarrow \pi^0\pi^0$ , calculated at threshold, in the presence of electromagnetic interactions and from which singularities of the infra-red photons have been appropriately subtracted [10]; the factor  $\gamma$  represents contributions at second-order of perturbation theory with respect to the nonrelativistic zeroth-order Coulomb hamiltonian of the bound state formalism. The explicit expressions

of  $\mathcal{R}e\widetilde{\mathcal{M}}_{00,+}$  and of  $\gamma$  may differ from one approach to the other, but their total contribution should be the same.

Our aim is to extend the previous analysis to the case of generalized  $\chi PT$  [11,4]. The latter is based on the observation that the fundamental order parameter of spontaneous chiral symmetry breaking is  $F_\pi$ , the decay coupling constant of the pion, which is related to the two-point function of left- and right-handed currents in the chiral limit. The other order parameters, such as the quark condensate in the chiral limit,  $\langle 0|\bar{q}q|0\rangle_0$ , have values depending on the details of the mechanism of chiral symmetry breaking and require independent experimental tests. Standard  $\chi PT$  is based on the assumption that the value of the Gell-Mann–Oakes–Renner (GOR) parameter [12], defined as

$$x_{GOR} = -\frac{2\hat{m}\langle 0|\bar{q}q|0\rangle_0}{F_\pi^2 m_\pi^2}, \quad (3)$$

where  $2\hat{m} = m_u + m_d$  and  $m_\pi$  is the pion physical mass, is close to one. Stated differently, the quark condensate parameter

$$B_0 \equiv -\frac{\langle 0|\bar{q}q|0\rangle_0}{F^2}, \quad (4)$$

where  $F$  is  $F_\pi$  in the chiral  $SU(2) \times SU(2)$  limit, is of the order of the hadronic mass scale  $\Lambda_H \sim 1$  GeV. This assumption fixes the way standard  $\chi PT$  is expanded: the quark condensate parameter  $B_0$  is assigned dimension zero in the infra-red external momenta of the Goldstone bosons, while the quark masses are assigned dimension two [2].

Generalized  $\chi PT$  relaxes the previous assumption and treats the order of magnitude of the quark condensate parameter  $B_0$  as an *a priori* unknown quantity (awaiting a precise experimental information about it) leaving to it the possibility of reaching small or vanishing values. To this aim,  $B_0$  is assigned dimension one in the infra-red momenta of the external Goldstone bosons and accordingly quark masses are also assigned dimension one. Due to this rule, at each order of the perturbative expansion, generalized  $\chi PT$  contains more terms than standard  $\chi PT$ . For instance, the pion mass formula becomes, at leading order,  $m_\pi^2 = 2\hat{m}B_0 + 4\hat{m}^2 A_0$ , where the constant

$A_0$  is expressible in terms of two-point functions of scalar and pseudoscalar quark densities. In the standard  $\chi PT$  case, this term is relegated to the next-to-leading order.

At the tree level of the chiral effective lagrangian, the  $\pi\pi$  scattering amplitude  $A(s|t, u)$  has the expression  $A(s|t, u) = (s - 2\hat{m}B_0)/F^2$ , which displays explicit dependence on the quark condensate parameter. It is useful to introduce two parameters,  $\alpha$  and  $\beta$ , that allow one to express the amplitude  $A$  in terms of the physical constants  $F_\pi$  and  $m_\pi$  [11,4]:

$$A(s|t, u) = \frac{\beta}{F_\pi^2} \left( s - \frac{4}{3} m_\pi^2 \right) + \alpha \frac{m_\pi^2}{3F_\pi^2}. \quad (5)$$

At leading order,  $F_\pi = F$  and  $\beta = 1$ . The GOR parameter (3) is then related to the parameter  $\alpha$  by the relation:  $x_{GOR} = (4 - \alpha)/3$ . For  $\alpha = 1$ , one recovers the standard  $\chi PT$  case, where  $x_{GOR} = 1$ . When  $\alpha$  increases,  $x_{GOR}$  decreases, corresponding to decreasing values of the quark condensate. For  $\alpha = 4$ , one reaches the extreme case of generalized  $\chi PT$ , where  $x_{GOR}$  and the quark condensate vanish. At higher orders, the relationship between  $\alpha$  and  $x_{GOR}$  becomes more complicated, involving the parameter  $\beta$  and other low energy constants, but the main qualitative feature found above remains: values of  $\alpha$  close to 1 correspond to the case of standard  $\chi PT$ , while large values of  $\alpha$ , above 1.5-2, say, cover the complementary domain of generalized  $\chi PT$ .

At two-loop order, the  $\pi\pi$  scattering amplitude is described by six parameters,  $\alpha$ ,  $\beta$ ,  $\lambda_1$ ,  $\lambda_2$ ,  $\lambda_3$  and  $\lambda_4$ . The five parameters other than  $\alpha$  are weakly dependent on the quark condensate and the values of the  $\lambda$ 's are fixed by a detailed analysis of sum rules using available data on  $\pi\pi$  scattering at medium energies. The parameter  $\beta$  is essentially sensitive to the deviation of  $F_\pi$  from  $F$  and remains in general close to 1. One ends up, in generalized  $\chi PT$ , with a complete determination of the threshold parameters of the  $\pi\pi$  scattering amplitude in terms of the possible numerical values of the parameter  $\alpha$  together with a best value of the other parameters. The expressions of the scattering lengths to two-loop order, as well as the values of the parameters  $\lambda$ , can be found in Ref. [4].

The evaluation of the corrections contained in the decay width formula (2) can be done in much the same way as in the standard  $\chi PT$  case, with the difference that whenever scattering lengths or the quark condensate parameter appear, these should be expressed through the generalized  $\chi PT$  formulas or parametrizations. We have followed the same method of approach as in Refs. [6,7] and skip here the details of the calculations. We only note that we made the approximation of taking the same values for the electromagnetic low energy constants  $k_i$  in generalized and standard  $\chi PT$  [13,14,10]. Actually, the effective lagrangian with electromagnetism is not yet available in generalized  $\chi PT$ , where the number of such constants should be larger. However, the contributions of the constants  $k_i$  not being dominant in the total decay width correction, such an approximation seems to be justified. Furthermore, a 100% uncertainty has been assigned to their global contributions in the various types of correction, their values being taken from Ref. [15]. Also, weakly dependent terms on the parameter  $\alpha$  have been numerically taken the same as in the standard case. Numerical estimates have been done with the following inputs:  $F_\pi = 92.4$  MeV,  $m_{\pi^+} = 139.57$  MeV,  $m_{\pi^0} = 134.97$  MeV.

We have chosen to analyze the total correction in terms of the scattering lengths, which are the physical quantities of interest, rather than of the intermediate parameters  $\alpha$  and  $\beta$ . Since the pionium lifetime concerns at leading order the combination  $(a_0^0 - a_0^2)$  of the scattering lengths, it is natural to promote this quantity as the main variable of the problem. As a second variable, we choose the  $P$ -wave scattering length, which is most sensitive to the parameter  $\beta$  and almost insensitive to  $\alpha$ . One then expresses the total correction in terms of  $(a_0^0 - a_0^2)$ ,  $a_1^1$  and the  $\lambda$ 's. (It is sufficient for the correction term to use the one-loop expressions of the scattering lengths, from which  $\lambda_3$  and  $\lambda_4$  are absent.) The experimental value of  $a_1^1 m_\pi^2$  is  $0.038 \pm 0.002$  [16]. One can then study the dependence of the decay width (or of the lifetime) upon the combination  $(a_0^0 - a_0^2)$  of the scattering lengths for fixed values of the above parameters.

The previous analysis can be repeated with var-

ious sets of inputs for  $a_1^1$  and for the  $\lambda$ 's within the intervals of their possible values. Generally, for a given  $(a_0^0 - a_0^2)$ , the lifetime  $\tau$  varies little for such changes of the inputs. The variations of the  $\lambda$ 's induce at most a variation of 0.8% on  $\tau$ , while the variations of  $a_1^1$  induce at most a variation of 1.1%. Assuming these variations as being uncorrelated, one may consider that 2% is an upper bound for the possible variations of  $\tau$ . Considering the latter as an uncertainty and adding it to the 2% uncertainty obtained in the course of calculation of  $\Delta\tau/\tau_0$ , one obtains a total uncertainty of 4% around the value of  $\tau$  calculated with the central values of  $a_1^1$  and of the  $\lambda$ 's.

The above analyzes are graphically summarized in Fig. 1. The full line represents the lifetime  $\tau$  as a function of the combination  $(a_0^0 - a_0^2)$ , corresponding to the central values of  $a_1^1$  and of the  $\lambda$ 's. The band around it corresponds to the estimated 4% total uncertainty. As already mentioned, the parameter  $\beta$  is almost insensitive to the variations of  $\alpha$ ; for instance, on the central line, it varies between 1.10 and 1.11. For a given experimental value of the lifetime, with a possible uncertainty, one can deduce from Fig. 1 the corresponding value of the combination  $(a_0^0 - a_0^2)$  with the related uncertainty.

The interpretation of the experimental value of the pionium lifetime will depend on its order of magnitude. We assume in the following discussion that, as the DIRAC experimental project foresees, the experimental uncertainty on the lifetime is of 10%. For larger uncertainties, the discussion should accordingly be modified. Three different possibilities may be considered. First, the central value of the lifetime is close to  $3 \times 10^{-15}$  s, lying above  $2.9 \times 10^{-15}$  s, say. Then standard  $\chi PT$  is firmly confirmed, since its predictions of  $(a_0^0 - a_0^2)$  lie between 0.250 and 0.258 [3]. Second, the central value of the lifetime lies below  $2.4 \times 10^{-15}$  s. Then it is the scheme of generalized  $\chi PT$  which is confirmed, since the corresponding central values of  $\alpha$  would lie above 2, which would mean that the quark condensate is not as large as assumed in the standard scheme. The third possibility is the most difficult to interpret. If the central value of the lifetime lies in the interval  $2.4 - 2.9 \times 10^{-15}$  s,

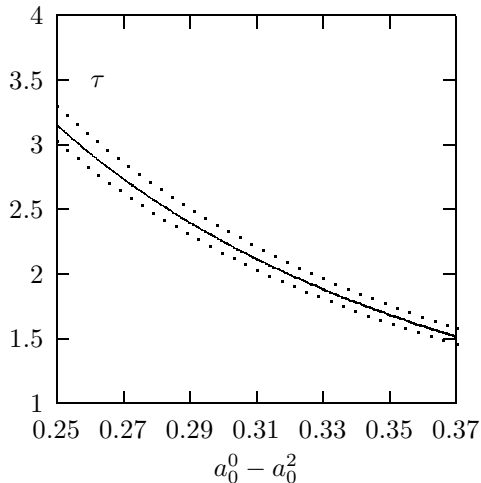


Figure 1. The pionium lifetime, in units of  $10^{-15}$  s, as a function of the combination  $(a_0^0 - a_0^2)$  of the  $S$ -wave scattering lengths. The full line corresponds to the central value  $a_1^1 m_\pi^2 = 0.038$  of the  $P$ -wave scattering length. The band delineated by the dotted lines takes account of the uncertainties, coming from theoretical evaluations, low energy constants and  $a_1^1$ .

then, because of the uncertainties, ambiguities in the interpretation may arise. Let us consider as an illustrative possibility the hypothetical result  $\tau = (2.60 \pm 0.26) \times 10^{-15}$  s. Without the inclusion of the uncertainties discussed above, this would imply  $(a_0^0 - a_0^2) = 0.278 \mp 0.015$  and  $\alpha = 1.68 \mp 0.42$ . Taking into account the 4% uncertainties represented by the band, one would have:  $(a_0^0 - a_0^2) = 0.278 \mp 0.022$  and  $\alpha = 1.68 \mp 0.61$ . Clearly, the upper bound of  $\tau$  is in favor of standard  $\chi PT$ , while its lower bound is in favor of generalized  $\chi PT$ . Such a situation would not disentangle the two schemes from each other. The uncertainties might be reduced in this case by a more refined analysis, taking into account the correlations between variations of  $a_1^1$  and of the  $\lambda$ 's.

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