Self-consistent parametrization of the two-flavor isotropic color-superconducting ground state

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Abstract

Lack of Lorentz invariance of QCD at finite quark chemical potential in general implies the need of Lorentz non-invariant condensates for the self-consistent description of the color-superconducting ground state. Moreover, the spontaneous breakdown of color $SU(3)$ in this state naturally leads to the existence of $SU(3)$ non-invariant non-superconducting expectation values. We illustrate these observations by analyzing the properties of an effective 2-flavor Nambu–Jona-Lasinio type Lagrangian and discuss the possibility of color-superconducting states with effectively gapless fermionic excitations. It turns out that the effect of condensates so far neglected can yield new interesting phenomena.

I. INTRODUCTION

According to current wisdom the deconfined 2-flavor QCD matter at moderate baryon densities and low temperatures behaves as a color superconductor [1]. Standard characteristics of its BCS-type ground state is the ground-state expectation value [2,3]

$$\delta_1 \;=\; \langle \psi^T \, C\gamma_5 \, \tau_2 \, \lambda_2 \, \psi \rangle \;, \tag{1}$$

which corresponds to a scalar diquark condensate in a color anti-triplet state. Here the superscript T denotes a transposition and C the matrix of charge conjugation. τ_2 is a Pauli matrix acting in flavor space, λ_2 a Gell-Mann matrix acting in color space. We have taken the freedom to rotate the unpaired quark color into the 3-direction, which can always be done.

Because of the empirical fact that the (approximate) chiral SU(2) symmetry of the QCD Lagrangian is not respected by the QCD vacuum, it is natural to ask whether the quark (-antiquark) condensate

$$\phi \;=\; \langle \bar{\psi} \, \psi \rangle \;, \tag{2}$$

persists also in the ground state of QCD matter at finite baryon density. For model Lagrangians with exact chiral symmetry one usually finds a first-order phase transition from a low-density phase (or vacuum) with $\phi \neq 0$ and $\delta_1 = 0$ to a high-density phase with $\delta_1 \neq 0$ and $\phi = 0$. This is different if there is a small quark mass m which explicitly breaks chiral symmetry. In this case ϕ cannot exactly vanish above the phase transition and coexists with the diquark condensate. In fact, just above the phase transition the gaps related to the two condensates can be of similar magnitude [4].

At finite density the existence of Lorentz non-invariant expectation values becomes possible. The most obvious example is of course the density itself,

$$\rho \;=\; \langle \bar{\psi} \, \gamma^0 \, \psi \rangle \;, \tag{3}$$

which transforms like the time component of a 4-vector. Consequently, if the Pauli principle and the form of the interaction permit, the ground state of a 2-flavor color superconductor is characterized by more condensates than merely the Lorentz-invariant ones, (1) and (2). This simple observation was made already in the pioneering work of Bailin and Love [5]. For instance, there could be another diquark condensate of the form [5–7]

$$\delta_2 \;=\; \langle \psi^T\, C\gamma^0\gamma_5\, \tau_2\, \lambda_2\, \psi \rangle \;, \tag{4}$$

which also transforms like the time component of a 4-vector. The ground state does not even have to be isotropic, but rotational invariance could be spontaneously broken, like in a ferromagnet. Even the breakdown of translational invariance due to the formation of crystalline phases is a possible scenario [8,9].

Up to now, we have discussed only space-time symmetries. We should recall, however, that in the presence of the diquark condensates (1) or (4) also color $SU(3)$ is spontaneously broken, since only the first two colors ("red" and "green") participate in the condensate, while the third one ("blue") does not. Therefore, there is no reason to assume that all other condensates are color-$SU(3)$ invariant in this state. For instance, we should expect that the contributions of red and blue quarks, ϕ_r and ϕ_b, to the quark condensate ϕ could be different, thus giving rise to a non-vanishing expectation value

$$\phi_8 \;=\; \langle \bar\psi\, \lambda_8\, \psi \rangle \;=\; \frac{2}{\sqrt{3}}\,(\phi_r - \phi_b) \;. \tag{5}$$

(Note that δ_1 and δ_2 leave a color $SU(2)$ subgroup invariant and therefore all green quantities are identical to the red ones). Similarly, the densities of red and blue quarks will in general not be the same, i.e., in addition to the total number density $\rho = 2\rho_r + \rho_b$ there could be a non-vanishing expectation value

$$\rho_8 \;=\; \langle \bar\psi\, \gamma^0\, \lambda_8\, \psi \rangle \;=\; \frac{2}{\sqrt{3}}\,(\rho_r - \rho_b) \;. \tag{6}$$

Since these color-symmetry breaking expectation values, induced by the presence of color-symmetry breaking diquark condensates, could in turn influence the properties of the diquark condensates, in principle, all condensates should be studied in a self-consistent way.

Detailed understanding of the color-superconducting state which in Nature might exist in the interiors of neutron stars [10] requires, however, a detailed knowledge of the effective quark-quark interactions close to their Fermi surface [11]. Due to the lack of experimental data or information from the lattice such a knowledge is missing at present. It is nevertheless possible to analyze in detail the effective low-energy quantum field theory of the deconfined low (or zero) temperature quark matter within models which respect all relevant symmetries of the corresponding QCD Lagrangian. The concept is basically that of a relativistic Landau Fermi-liquid theory: Starting point is an effective Lagrangian $\mathcal{L}_{\rceil\{\{}$ which might in principle arise from the underlying microscopic theory (i.e., QCD) by resummation of a complicated infinite set of diagrams. $\mathcal{L}_{\rceil\{\{}$ itself is often non-renormalizable and only meaningful together with an approximation scheme which has to be specified in advance. In our case (and in most other papers on this subject, see e.g. [1–4], [6]), the thermodynamic potential is derived in Hartree-Fock-Bogoliubov (self-consistent one-loop) approximation. Being variational in character it is generally assumed to be justified also for strong couplings.

Lack of Lorentz invariance of QCD at finite μ implies that $\mathcal{L}_{eff}$ itself should respect only an $O(3)$ rotational symmetry. Since there are no thermal gluons at $T = 0$ it is justified and customary to analyze the ground-state properties of deconfined quark matter by virtue of $\mathcal{L}_{eff}$ having only a global color $SU(3)$ symmetry. Response of gluons to the quark condensates carrying color (Meissner effect) is then studied [12,13] perturbatively, i.e., as if the color gauge symmetry is spontaneously broken (for more detailed discussion see [1]). In this article we illustrate the need of Lorentz- and color-$SU(3)$ non-invariant condensates for the self-consistent description of the color-superconducting ground state. For simplicity we leave the rotational symmetry unbroken, i.e., we restrict ourselves to the isotropic case.

The remainder of this article is organized as follows. In Sec. II we will derive the thermodynamic potential and a coupled set of gap equations for a general NJL-type model, taking into account a possible condensation in the channels (1) to (6). The resulting dispersion laws are discussed in more detail in Sec. III. Here special emphasis is put on possible gapless color superconductors. In Sec. IV we discuss our numerical results. Conclusions are drawn

in Sec. V.

II. GENERAL FORMALISM

In order to illustrate the generic properties of the superconducting ground state let us -as an example- consider a model defined by the Lagrangian density

$$\mathcal{L}_{eff} \;=\; \bar{\psi}(i\partial\!\!\!/ - m)\psi \;+\; \mathcal{L}_{q\bar{q}} \;+\; \mathcal{L}_{qq} \tag{7}$$

describing a quark of mass m which interacts with other quarks and antiquarks via NJL-type 4-point interactions of the color SU(3) invariant form

$$\mathcal{L}_{q\bar{q}} \;=\; g_s^{(0)}\,(\bar{\psi}\psi)^2 \;+\; g_s^{(8)}\sum_{a=1}^{8}(\bar{\psi}\lambda_a\psi)^2 \;+\; g_v^{(0)}\,(\bar{\psi}\gamma^0\psi)^2 \;+\; g_v^{(8)}\sum_{a=1}^{8}(\bar{\psi}\gamma^0\lambda_a\psi)^2 \;+\; ... \tag{8}$$

and

$$\mathcal{L}_{qq} \;=\; h_1\sum_{A=2,5,7}(\bar{\psi}\,i\gamma_5\tau_2\lambda_A\,\psi^c)(\bar{\psi}^c\,i\gamma_5\tau_2\lambda_A\,\psi) \;+\; h_2\sum_{A=2,5,7}(\bar{\psi}\,\gamma^0\gamma_5\tau_2\lambda_A\,\psi^c)(\bar{\psi}^c\,\gamma^0\gamma_5\tau_2\lambda_A\,\psi) \;+\; ... \tag{9}$$

Here the dots indicate possible other channels, not related to the expectation values (1) to (6). In Eq. (9) we used the notation $\psi^c = C\bar{\psi}^T$, while λ_A denotes the antisymmetric Gell-Mann matrices λ_2, λ_5, and λ_7. All color indices are understood to be summed over.

In the following it is convenient to formally double the degrees of freedom by defining

$$q(x) \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix}\psi(x)\\ \psi^c(x)\end{pmatrix}\;. \tag{10}$$

To obtain the mean-field thermodynamic potential at temperature T and chemical potential μ we linearize $\mathcal{L}_{eff}$ in the vicinity of the expectation values Eqs. (1) to (6) and apply Matsubara formalism. The resulting thermodynamic potential per volume reads

$$\Omega(T,\mu) \;=\; -T\sum_n\int\frac{d^3p}{(2\pi)^3}\;\frac{1}{2}\,\mathrm{Tr}\;\ln\left(\frac{1}{T}\,S^{-1}(i\omega_n,\vec{p})\right) \;+\; V\;, \tag{11}$$

with ω_n being fermionic Matsubara frequencies. $S^{-1}(p)$ is the inverse propagator of the q-fields at 4-momentum p. It is given by

$$S^{-1}(p) \;=\; \begin{pmatrix} \not{p} - M_0 - M_8\lambda_8 + \mu_0\gamma^0 + \mu_8\gamma^0\lambda_8 & \Delta_1\gamma_5\tau_2\lambda_2 + \Delta_2\gamma^0\gamma_5\tau_2\lambda_2 \\ -\Delta_1^*\gamma_5\tau_2\lambda_2 + \Delta_2^*\gamma^0\gamma_5\tau_2\lambda_2 & \not{p} - M_0 - M_8\lambda_8 - \mu_0\gamma^0 - \mu_8\gamma^0\lambda_8 \end{pmatrix} . \quad (12)$$

Here we introduced the effective quark masses

$$M_0 \;=\; m - 2g_s^{(0)}\phi \;, \qquad M_8 \;=\; -2g_s^{(8)}\phi_8 \;, \quad (13)$$

the effective chemical potentials

$$\mu_0 \;=\; \mu + 2g_v^{(0)}\rho \;, \qquad \mu_8 \;=\; 2g_v^{(8)}\rho_8 \;, \quad (14)$$

and the diquark gaps

$$\Delta_1 \;=\; -2h_1\delta_1 \;, \qquad \Delta_2 \;=\; 2h_2\delta_2 \;. \quad (15)$$

These quantities also enter into the last term of Eq. (11), which is defined as

$$V \;=\; \frac{(M_0-m)^2}{4g_s^{(0)}} \;+\; \frac{M_8^2}{4g_s^{(8)}} \;+\; \frac{(\mu_0-\mu)^2}{4g_v^{(0)}} \;+\; \frac{\mu_8^2}{4g_v^{(8)}} \;+\; \frac{|\Delta_1|^2}{4h_1} \;+\; \frac{|\Delta_2|^2}{4h_2} \;. \quad (16)$$

For later convenience, but also for the interpretation of the results, it is useful to perform linear combinations to get red and blue quantities, e.g. red and blue constituent quark masses $M_r = M_0 + \frac{1}{\sqrt{3}}\,M_8$ and $M_b = M_0 - \frac{2}{\sqrt{3}}\,M_8$. We then find

$$\begin{aligned} M_r \;&=\; m - \frac{2}{3}(6g_s^{(0)} + 2g_s^{(8)})\phi_r \;-\; \frac{2}{3}(3g_s^{(0)} - 2g_s^{(8)})\phi_b \,, \\ M_b \;&=\; m - \frac{2}{3}(6g_s^{(0)} - 4g_s^{(8)})\phi_r \;-\; \frac{2}{3}(3g_s^{(0)} + 4g_s^{(8)})\phi_b \,, \\ \mu_r \;&=\; \mu \;+\; \frac{2}{3}(6g_v^{(0)} + 2g_v^{(8)})\rho_r \;+\; \frac{2}{3}(3g_v^{(0)} - 2g_v^{(8)})\rho_b \,, \\ \mu_b \;&=\; \mu \;+\; \frac{2}{3}(6g_v^{(0)} - 4g_v^{(8)})\rho_r \;+\; \frac{2}{3}(3g_v^{(0)} + 4g_v^{(8)})\rho_b \,. \end{aligned} \quad (17)$$

For two flavors and three colors the inverse propagator, Eq. (12), is a 48×48 matrix, and the trace in Eq. (11) has to be taken in this 48-dimensional space. After performing the Matsubara sum we obtain:

$$\begin{aligned} \Omega(T,\mu) \;=\; -4\int \frac{d^3p}{(2\pi)^3} \Big\{ \quad & 2 \quad \Big(\tfrac{E_+ + E_-}{2} + T\,\ln(1+e^{-E_+/T}) \;+\; T\,\ln(1+e^{-E_-/T})\Big) \\ & + \quad \Big(\quad \epsilon_b \;+ T\,\ln(1+e^{-\epsilon_+/T}) \;+\; T\,\ln(1+e^{-\epsilon_-/T})\Big)\Big\} \\ + V \,, & \end{aligned} \quad (18)$$

where physically irrelevant constant terms have been suppressed. The dispersion laws which enter into this expression are given by

$$\epsilon_\pm \;=\; \epsilon_b \pm \mu_b \;=\; \sqrt{\vec{p}^2 + M_b^2} \pm \mu_b \tag{19}$$

and

$$E_\pm \;=\; \sqrt{\vec{p}^2 + M_r^2 + \mu_r^2 + |\Delta_1|^2 + |\Delta_2|^2 \pm 2s}\,, \tag{20}$$

with

$$s \;=\; \sqrt{(\mu_r^2 + |\Delta_2|^2)\vec{p}^2 + t^2}\,, \qquad t \;=\; M_r\mu_r - Re(\Delta_1\Delta_2^*)\,. \tag{21}$$

So far, the thermodynamic potential depends on our choice of the expectation values (1) to (6), which determine the effective masses, effective chemical potentials and diquark gaps as indicated above. On the other hand, in a thermodynamically consistent treatment the condensates should follow from the thermodynamic potential by taking the appropriate derivatives. The self-consistent solutions are given by the stationary points of the potential,

$$\frac{\delta\Omega}{\delta M_0} \;=\; \frac{\delta\Omega}{\delta M_8} \;=\; \frac{\delta\Omega}{\delta\mu_0} \;=\; \frac{\delta\Omega}{\delta\mu_8} \;=\; \frac{\delta\Omega}{\delta\Delta_1} \;=\; \frac{\delta\Omega}{\delta\Delta_2} \;=\; 0\,. \tag{22}$$

If there is more than one stationary point, the stable solution is the one which corresponds to the lowest value of $\Omega(T,\mu)$.

Eqs. (18) and (22) lead to the following expressions for the various expectation values:

$$\begin{aligned}
\phi_r \;&=\; -4\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{2s}\,\Big\{\quad (1-2n(E_+))\,\frac{1}{E_+}\,[M_r s + \mu_r t] \\
&\qquad\qquad + (1-2n(E_-))\,\frac{1}{E_-}\,[M_r s - \mu_r t]\;\Big\}\,, \\
\phi_b \;&=\; -4\int\frac{d^3p}{(2\pi)^3}\,\frac{M_b}{\epsilon_b}\,(1-n(\epsilon_+)-n(\epsilon_-))\,, \\
\rho_r \;&=\; 4\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{2s}\,\Big\{\quad (1-2n(E_+))\,\frac{1}{E_+}\,[\mu_r(s+\vec{p}^2) + M_r t] \\
&\qquad\qquad + (1-2n(E_-))\,\frac{1}{E_-}\,[\mu_r(s-\vec{p}^2) - M_r t]\;\Big\}\,, \\
\rho_b \;&=\; 4\int\frac{d^3p}{(2\pi)^3}\,(n(\epsilon_-)-n(\epsilon_+))\,,
\end{aligned}$$

$$
\begin{aligned}
\delta_1 &= -4\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{s}\,\Big\{\quad (1-2n(E_+))\,\frac{1}{E_+}\,[\Delta_1 s-\Delta_2 t] \\
&\qquad\qquad + (1-2n(E_-))\,\frac{1}{E_-}\,[\Delta_1 s+\Delta_2 t]\,\Big\}\,, \\
\delta_2 &= 4\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{s}\,\Big\{\quad (1-2n(E_+))\,\frac{1}{E_+}\,[\Delta_2(s+\vec{p}^{\,2})-\Delta_1 t] \\
&\qquad\qquad + (1-2n(E_-))\,\frac{1}{E_-}\,[\Delta_2(s-\vec{p}^{\,2})+\Delta_1 t]\,\Big\}\,,
\end{aligned}
\tag{23}
$$

where $n(E) = 1/(e^{E/T}+1)$ is a Fermi function. Together with Eqs. (15) and (17) these equations form a set of six coupled gap equations for M_r, M_b, μ_r, μ_b, Δ_1 and Δ_2. The expressions for the blue expectation values ϕ_b and ρ_b formally look like the corresponding formulae for free particles. However, they depend on the effective quantities M_b and μ_b, which are also related to red quantities via Eq. (17). Despite of these interdependencies, the masses of red and blue quarks, and also their densities will in general be different, as anticipated above.

Another interesting observation is that in general δ_1 and δ_2 or, equivalently, Δ_1 and Δ_2 cannot vanish separately. This means that the familiar scalar diquark condensate δ_1 will in general be accompanied by an induced non-vanishing expectation value δ_2. [1] On the other hand, there is always a solution with $\Delta_1 = \Delta_2 = 0$. In this case the expressions for ϕ_r and ρ_r get the same structure as the analogous expressions for the blue quarks. If Δ_1 and Δ_2 do not vanish, they are determined by the gap equation only up to a common phase. In most cases this phase can be chosen such that Δ_1 and Δ_2 are real.

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[1] This point has already been discussed in Ref. [7]. The authors, however, argued that because of the small value of δ_2 as compared with δ_1, this condensate could for simplicity be neglected although strictly speaking in this case the system of gap equations cannot be closed.

III. DISPERSION LAWS AND GAPLESS COLOR-SUPERCONDUCTING STATES

In physical terms the derived mean-field thermodynamic potential, Eq. (18), describes the thermodynamics of a mixture of noninteracting fermionic excitations of two types: (A) quark excitations with the dispersion law $\epsilon_{\pm}$ and (B) Bogoliubov-Valatin (BV) quasiquark excitations with the dispersion law $E_{\pm}$.[2] Straightforward interpretation is to say that these dispersion laws should be used for calculating (in principle) measurable quantities, e.g. the specific heat of the system. The parameters entering the dispersion laws, i.e., M_b, M_r, μ_b, μ_r, $|\Delta_1|$, $|\Delta_2|$, and $\cos(\varphi_1 - \varphi_2)$, $(\Delta_j = |\Delta_j| \exp(i\varphi_j), j = 1, 2)$ are fixed as the solutions of the coupled nonlinear integral equations (23) in terms of m, μ, and the dimensional coupling constant(s) of a given four-fermion interaction. Due to the fact that the integrals in Eq. (23) have to be regularized there is one more dimensionful parameter upon which the above quantities depend. This can be a cutoff, a parameter in a form-factor [2], the instanton size [14], or a thickness of a layer around the Fermi surface in which the interaction is assumed to be different from zero [3].

Component (A) of the system behaves like a normal relativistic Fermi gas, characterized by a linear dependence of the low-temperature specific heat on T. The actual behavior of the physically interesting BV component of the system characterized by $E_{-}(\vec{p})$ depends strongly upon the details of the interaction. For instance, the instanton mediated Lagrangians employed in Refs. [2] and [3], [14] have $h_2 = 0$ and therefore $\Delta_2 = 0$. In this case Eq. (20) reduces to the "classic" result

$$E_{-}(\vec{p}) = \sqrt{(\sqrt{\vec{p}^2 + M_r^2} - \mu_r)^2 + |\Delta_1|^2}\,. \tag{24}$$

There are no nodes in this function, and this component exhibits a superconducting behavior

[2]Strictly speaking, the quarks of component (A) are also quasiparticles as long as M_b is different from the bare mass m.

including the exponential low-T specific heat characteristic of ordinary superconductors.

In the general case the form of $E_-(\vec{p})$ is given by Eq. (20). Clearly, without Lorentz invariance the dependence of the energy of a particle-like excitation on momentum is restricted only by positivity. It is not surprising, however, that $E_-(\vec{p})$ can be parametrized as

$$E_-(\vec{p}) \equiv \sqrt{(\sqrt{\vec{p}^2 + M_{eff}^2} - \mu_{eff})^2 + |\Delta_{eff}|^2}\,, \tag{25}$$

provided we identify

$$\mu_{eff} \equiv \sqrt{\mu_r^2 + |\Delta_2|^2}\,, \quad M_{eff}^2 \equiv \frac{(M_r\mu_r - Re\Delta_1\Delta_2^*)^2}{\mu_{eff}^2}\,, \tag{26}$$

and

$$M_r^2 + |\Delta_1|^2 \equiv M_{eff}^2 + |\Delta_{eff}|^2\,. \tag{27}$$

It is interesting to notice that $|\Delta_{eff}|$ can vanish, i.e., $E_-(\vec{p})$ can have nodes at $\vec{p}^2 = \mu_{eff}^2 - M_{eff}^2$. This is the case if

$$M_r\Delta_2 = -\mu_r\Delta_1\ . \tag{28}$$

If $\mu_r \neq 0$ this relation can be used to eliminate Δ_1 in Eq. (26):

$$\mu_{eff}^2 = \mu_r^2\,(1 + \frac{|\Delta_2|^2}{\mu_r^2})\,, \quad M_{eff}^2 = M_r^2\,(1 + \frac{|\Delta_2|^2}{\mu_r^2})\,, \tag{29}$$

and hence

$$\vec{p}^2_{node} = (\mu_r^2 - M_r^2)\,(1 + \frac{|\Delta_2|^2}{\mu_r^2})\,. \tag{30}$$

This means, μ_r^2 must be greater or equal to M_r^2 and it immediately follows from Eq. (28) that a gapless color-superconducting solution is only possible if $|\Delta_2| \geq |\Delta_1|$.

In the vicinity of the node the BV quasiparticle takes the form of a non-relativistic fermion,

$$E_-(\vec{p}) \approx \frac{\vec{p}^2}{2m^*} - \mu^*\,, \tag{31}$$

where $m^* = \mu_{eff}$ and $\mu^* = \frac{1}{2}(\mu_{eff} - \frac{M_{eff}^2}{\mu_{eff}})$. Despite the superconducting condensates the specific heat of such a system is linear in T. A similar phenomenon was found in Ref. [15] in the color-flavor locked (CFL) phase, though its origin is not the same as here.

In the above discussion it is tacitly assumed that there is an interaction which yields solutions of the coupled gap equations for which Eq. (28) holds. A particularly simple way to fulfill this condition is to assume that $M_r = \Delta_1 = 0$, which could be realized by taking $m = g_i^{(k)} = h_1 = 0$ and only $h_2 \neq 0$. If we regularize the divergent integrals by a sharp 3-momentum cut-off Λ and restrict ourselves to $T = 0$, the thermodynamic potential of this schematic model is readily calculated:

$$\Omega_{schem}(T=0,\mu;\Delta_2) \;=\; -\frac{1}{6\pi^2}\Big(\; 2(\mu^2+|\Delta_2|^2)^2+\mu^4 \;\Big) \;+\; \frac{|\Delta_2|^2}{4h_2}\,. \tag{32}$$

Here we dropped an irrelevant constant $-\frac{3}{2\pi^2}\Lambda^4$. Obviously this function is not bounded from below, but only the self-consistent solutions of the gap equation, i.e. $\delta\Omega_{schem}/\delta\Delta_2 = 0$, are physically meaningful. There is always a trivial solution with $\Delta_2 = 0$. For $0 < h_2 < \frac{3\pi^2}{8\mu^2}$ there are also nontrivial solutions with $|\Delta_2|^2 = \frac{3\pi^2}{8h_2} - \mu^2$. However, whenever these nontrivial solutions exist, they correspond to maxima of Ω_{schem}, while at same time the trivial solution is a local minimum with a lower value of Ω_{schem}. This means, the nontrivial gapless solution is unstable. Although we have shown this only for our very simple schematic model, it might be a rather general feature. In fact, a similar observation was made by the authors of Ref. [15] for the CFL phase. We will come back to this point in the end of the next section.

IV. NUMERICAL RESULTS

In the schematic example discussed in the end of the previous section the problem was reduced to a single condensate. In this section we want to present the results of a numerical study of the full coupled set of gap equations derived in Sec. II. As an example we consider an interaction with the structure of a "heavy-gluon exchange",

$$\mathcal{L}_{hge} \;=\; \bar\psi(i\partial\!\!\!/ - m)\psi \;-\; g_E\sum_{a=1}^{8}(\bar\psi\gamma^0\lambda_a\psi)^2 \;+\; g_M\sum_{a=1}^{8}(\bar\psi\vec\gamma\lambda_a\psi)^2\,, \tag{33}$$

although there is no reason why the effective interaction at moderate densities should have this particular form. As discussed in the introduction, the effective Lagrangian at finite densities does not need to be Lorentz invariant. We underline this possibility by explicitly allowing for different "electric" and "magnetic" coupling constants, g_E and g_M.

The effective quark-antiquark interaction $\mathcal{L}_{q\bar{q}}$ and the effective quark-quark interaction $\mathcal{L}_{qq}$ as given in Eqs. (8) and (9) can be derived from $\mathcal{L}_{hge}$ by performing the appropriate Fierz transformations. The resulting coupling constants are

$$\begin{aligned} g_s^{(0)} &= \tfrac{2}{9}\,(g_E + 3g_M)\;, \quad g_s^{(8)} = -\frac{1}{24}\,(g_E + 3g_M)\;, & h_1 &= \frac{1}{6}\,(g_E + 3g_M)\;, \\ g_v^{(0)} &= \tfrac{2}{9}\,(g_E - 3g_M)\;, \quad g_v^{(8)} = -g_E - \frac{1}{24}\,(g_E - 3g_M)\;, & h_2 &= \frac{1}{6}\,(g_E - 3g_M)\;. \end{aligned} \tag{34}$$

A. Lorentz invariant interaction

We begin our discussion with the case of a Lorentz-invariant interaction,

$$g \;\equiv\; g_E \;=\; g_M\,. \tag{35}$$

Although there is no reason for this assumption, it allows for a better comparison with other calculations in the literature, which often start from an interaction of this form. Of course, having a Lorentz-invariant interaction does not mean that there are only Lorentz-invariant condensates, since Lorentz-invariance is still broken by the chemical potential.

If we insert Eq. (35) into Eq. (34) we find that for $g > 0$ the interaction is attractive in the scalar quark-antiquark channel and in the scalar diquark channel and repulsive in all other channels of interest. Of course, non-vanishing expectation values in the repulsive channels do not develop spontaneously, but only as a result of an external source, like the chemical potential, or induced by non-vanishing expectation values in attractive channels. In fact, the solutions of the gap equations correspond to maxima of the thermodynamic potential with respect to variations in the repulsive channels, whereas they can be maxima or minima with respect to variations in the attractive channels.

In our numerical calculations we restrict ourselves again to $T = 0$ and take a sharp 3-momentum cut-off Λ to regularize the integrals. Although a quark model of the present type should be used only in the deconfined phase, we follow the general custom and fix the parameters such that they yield "reasonable" vacuum properties. In the following we take $\Lambda = 600$ MeV, $g\Lambda^2 = 2.75$, and a bare quark mass $m = 5$ MeV. With these parameter values we obtain a vacuum constituent quark mass $M_r = M_b = 407.7$ MeV. This corresponds to a quark condensate $\phi = -2(245.7\text{MeV})^3$, while ϕ_8, ρ, ρ_8, δ_1 and δ_2 vanish in vacuum.

When we increase the quark chemical potential nothing happens up to a critical value $\mu_{crit} = 403.3$ MeV. At this point a first-order phase transition takes place and all expectation values Eqs. (1) to (6) receive non-vanishing values. This can be inferred from Fig. 1 where various quantities are displayed as functions of μ. In the left panel we show the constituent quark mass M_r, the diquark gap Δ_1, and the effective chemical potentials μ_r and μ_b. In the right panel the mass difference $M_r - M_b$ and the diquark gap $-\Delta_2$ are plotted. At $\mu = \mu_{crit}$ the constituent quark masses drop by more than 300 MeV and are no longer identical. The difference, however, is small, $M_r = 95.7$ MeV and $M_b = 95.0$ MeV. With increasing chemical potential, both, the masses and their difference, are further decreased. In the diquark channel we find $\Delta_1 = 120.0$ MeV at $\mu = \mu_{crit}$. Similar to what has been found in Ref. [7], the second gap parameter has the opposite sign and is more than one order of magnitude smaller, $\Delta_2 = $ -8.4 MeV. Like the constituent masses, it decreases with increasing μ, whereas Δ_1 is slightly growing in the regime shown in Fig. 1.

Below the phase transition, the effective chemical potentials μ_r and μ_b are equal to the external chemical potential μ. At the phase transition point, μ_r and μ_b drop by 67 MeV and 51 MeV, respectively and then grow again as functions of μ. The corresponding number densities of red and blue quarks are shown in Fig. 2. At $\mu = \mu_{crit}$, ρ_r jumps from zero to 0.42 fm^{-3}, about 2.5 times nuclear matter density, whereas the density of blue quarks reaches only twice nuclear matter density at this point, $\rho_b = 0.34$ fm^{-3}. Both densities grow of course with increasing chemical potential, but their difference remains nearly constant.

The unequal densities of red and blue quarks in this state, although anticipated, clearly

deserve further discussion. First of all we should point out that $\rho_r - \rho_b \neq 0$ is not a result of our self-consistent treatment, but is already found if we only take into account the effect of a non-vanishing Δ_1. Nevertheless, it is usually not discussed in the literature.

Since our model Lagrangian $\mathcal{L}_{hge}$ is symmetric under global color $SU(3)$ transformations, the total number of quarks of each color is a conserved quantity. As in BCS theory, quark number conservation is violated by the diquark condensates, but the average numbers are still conserved and given by the densities ρ_i integrated over the volume. One could therefore ask what happens if we start with a large but finite system with equal numbers of red, green and blue quarks at low densities and then compress it until a color anti-triplet diquark condensate is formed. According to the above results, in this phase there are not equally many quarks of each color, but the number of those quarks which do not participate in the condensate (in the above case the blue ones) is smaller. Obviously, we cannot get a single phase with these properties if we start from a system with equal densities. A possible scenario could be that several domains emerge in which the symmetry is broken into different directions, such that the total number of red, green and blue quarks remains unchanged. Still one could worry about strong forces inside these domains arising from the non-vanishing net color charge of the domain. However, we should keep in mind that by construction our quarks are non-interacting quasiparticles, i.e., at least on the mean field level there are no forces left.

Alternatively, we can construct a superconducting state with equal densities for the gapped and ungapped quarks. To that end we have to introduce different *external* chemical potentials for different red and blue quarks, or, equivalently, an additional external chemical potential μ_8^{ext}. Then the second equation in Eq. (14) becomes

$$\mu_8 \;=\; \mu_8^{ext} + 2g_v^{(8)}\rho_8 \;, \tag{36}$$

and in Eq. (16) we should replace μ_8^2 by $(\mu_8 - \mu_8^{ext})^2$. With this additional external parameter we can enforce the densities of all colors to be equal, even in the superconducting state. Obviously this is the case if $\mu_8 = \mu_8^{ext}$.

However, at least within our mean field approximation such a state would be energetically less favored than a state with the same total density, but $\mu_8^{ext} = 0$. This is shown in the right panel of Fig. 2. The energy density ε of the system is given by

$$\varepsilon(T=0,\rho,\rho_8) \;=\; \Omega(T=0,\mu,\mu_8) - \Omega(0,0,0) \;+\; \mu\rho \;+\; \mu_8^{ext}\rho_8\,. \tag{37}$$

The second term on the r.h.s. has been introduced to normalize the energy density of the (non-trivial) vacuum to zero. In the right panel of Fig. 2 the energy per quark, ε/ρ, is displayed as a function of the total quark number density ρ. The solid line is the result for $\mu_8^{ext} = 0$, i.e., it corresponds to the unequal red and blue quark densities as shown in the left panel. The dashed line corresponds to the result for equal red and blue quark densities. As one can see in the figure, the energy of this solution is always higher than the energy of the solution with unequal densities. This means, according to this result, a large homogeneous system of equally many red, green and blue quarks is unstable against decay into several domains in which the density of the gapped quarks is larger than the density of the remaining ungapped quark. On the other hand, the energy difference is not very large (less than 1 MeV per quark). Therefore it cannot be excluded that the homogeneous solution with equal densities would be favored, once effects beyond the mean-field approximation are taken into account.

B. Lorentz non-invariant interaction

In the previous section we found rather small values for the gap parameter Δ_2. This seems to justify the common practice to neglect this condensate completely. In order to find out whether the smallness of Δ_2 is a general feature, we now want to abandon the unnecessary restriction to a Lorentz invariant interaction, Eq. (35), and to allow for unequal values of g_E and g_M. Looking at Eq. (34), we see that the coupling constants $g_s^{(0)}$, $g_s^{(8)}$ and h_1 depend on the sum $g_E + 3g_M$, whereas $g_v^{(0)}$ and h_2 depend on the difference $g_E - 3g_M$. $g_v^{(8)}$ is somewhat exceptional because it is mostly given by the "direct" interaction term,

which is not present in the other channels. The "exchange" term is again proportional to $g_E - 3g_M$, but suppressed by a factor 1/24.

In principle there are two ways to obtain larger values of Δ_2. The more dramatic one is to assume that g_E is larger than $3g_M$. Then the coupling constants $g_v^{(0)}$ and h_2 are positive, i.e., the interaction becomes attractive in these channels. Indeed, as we will see in Sec. IV C, an attractive h_2 can yield solutions with large values of Δ_2. However, these solutions might not be stable, similar to what we found for the schematic example in the end of Sec. III. Unfortunately, a detailed study of the structure of solutions of the coupled gap equations involving four attractive channels is very difficult and beyond the scope of our paper.

The second possibility is to make h_2 more repulsive. In order to have a similar starting point as in the previous section, we keep the sum $g_E + 3g_M$ fixed, but increase the difference $3g_M - g_E$. To see a relatively large effect we take $g_E = -1.75/\Lambda^2$ and $g_M = 4.25/\Lambda^2$, i.e., $3g_M - g_E = 14.5/\Lambda^2$, almost three times as large as in the previous section. As before, we take $m = 5$ MeV and a cutoff $\Lambda = 600$ MeV. Our results are displayed in Fig. 3. In the left panel we show the behavior of M_r, Δ_1, $-\Delta_2$, and μ_r as functions of μ. The most striking difference to our previous example (Fig. 1) is the fact that we now find a smooth crossover instead of a first-order phase transition. In the chiral limit the phase transition becomes second order. A similar effect is known from studies of the chiral phase transition within the NJL model without diquark condensation. There it was also found that a first-order phase transition becomes second order (or a smooth crossover) if the coupling strength in the (repulsive) vector channel exceeds a certain value [16]. Although the present model is rather unrealistic below $\mu \sim 450$ MeV, where it predicts a color-superconducting quark gas of low density, it is nevertheless a counter example to the common (and mostly model based) belief that the chiral phase transition at zero temperature and large μ should be first order (see e.g. Refs. [4,17]).

Despite the fact that $|h_2|$ is larger than h_1, the absolute value of Δ_2 remains always smaller than that of Δ_1. However, unlike in the previous section, Δ_2 can no longer be generally neglected. For instance, at $\mu = 450$ MeV, we find $M_r = 280.1$ MeV, $\mu_r = 382.3$ MeV,

$\Delta_1 = 83.2$ MeV, and $\Delta_2 = -37.8$ MeV. In terms of Eqs. (26) and (27) this corresponds to $\mu_{eff} = 384.1$ MeV, $M_{eff} = 286.9$ MeV, and $|\Delta_{eff}| = 55.2$ MeV. The resulting dispersion law $E_-(\vec{p})$ is shown in the right panel of Fig. 3 (solid line). At $|\vec{p}| = \sqrt{\mu_{eff}^2 - M_{eff}^2} = 255.4$ MeV it has a minimum with $E_- = |\Delta_{eff}| = 55.2$ MeV. On the other hand, if we neglect Δ_2 in the gap equation, we get $M_{eff} = M_r = 292.5$ MeV, $\mu_{eff} = \mu_r = 383.3$ MeV, and $\Delta_{eff} = \Delta_1 = 102.2$ MeV. Consequently, the minimum value of E_- is now 102.2 MeV, almost twice as much as without neglecting Δ_2. The corresponding dispersion law is indicated by the dashed line in the right panel of Fig. 3. As one can see, the entire function $E_-(\vec{p})$ is shifted to higher energies as compared with the solid curve and the minimum is more shallow.

C. Gapless color superconductors

Finally, we would like to come back to the question of possible gapless color superconductors. Obviously, none of our numerical examples presented so far came close to condition (28). For instance, if we take $M_r = 280.1$ MeV, $\mu_r = 382.3$ MeV, and $\Delta_1 = 83.2$ MeV, as found in the previous section at $\mu = 450$ MeV, one would need $\Delta_2 \simeq -113.6$ MeV, about three times as large as the actual value. The situation was even worse in Sec. IV A where the discrepancy was about a factor 50 at $\mu = \mu_{crit}$ and became larger with increasing chemical potential. In fact, none of our numerical examples fulfilled $|\Delta_2| \geq |\Delta_1|$, which we identified as a necessary condition for gapless color superconducting states.

To get some insight, how an interaction could look like which yields such a state, we can invert the problem and employ the gap equations to calculate the effective coupling constants which are consistent with a given set of gap parameters. For instance, Eq. (28) is obviously fulfilled if we choose $M_r = \Delta_1 = 100$ MeV and $\mu_r = -\Delta_2 = 350$ MeV. For simplicity we might assume $M_b = M_r$ and $\mu_b = \mu_r$. Except of Δ_2 this is within the typical range of these quantities in the earlier examples. If we now take $m = 5$ MeV and a cutoff $\Lambda = 600$ MeV, as before, and $\mu = 450$ MeV, the gap equations yield $g_s^{(0)}\Lambda^2 = 3.36$, $g_v^{(0)}\Lambda^2 = -1.41$,

$h_1\Lambda^2 = 6.80$, $h_2\Lambda^2 = 6.18$, and $g_s^{(8)}\Lambda^2 = g_v^{(8)}\Lambda^2 = 0$. Here the essential difference to our earlier examples is the need of an attractive interaction in the h_2 channel. Furthermore, the interaction is relatively strong in both diquark channels. However, for these parameters there is another solution with $M_r = M_b = 58.1$ MeV, $\mu_r = \mu_b = 362.8$ MeV, $\Delta_1 = 966.6$ MeV, and $\Delta_2 = 16.1$ MeV. In order to decide which of the two solutions is the energetically favored one we have to evaluate the thermodynamic potential. It turns out that for the gapless solution the value of Ω is about 900 MeV/fm^3 higher than for the other solution. Hence, similar to what we found in the schematic example of Sec. III, the gapless state does not correspond to a stable solution. In fact, we did not succeed to construct a stable color-superconducting solution. This might indicate that such a state does not exist, although a rigorous proof is still missing.

V. CONCLUSIONS

We analyzed the ground state properties of an isotropic two-flavor color-superconductor at finite quark chemical potential within an NJL-type model. In general, a self-consistent treatment of this problem requires to consider several condensates which are usually neglected: The breakdown of Lorentz invariance in dense systems implies the possible existence of Lorentz non-invariant diquark condensates, while the spontaneous breaking of color $SU(3)$ in a color superconductor naturally leads to the existence of $SU(3)$ non-invariant quark-antiquark condensates. We found that at least six different expectation values (two diquark condensates and four quark-antiquark condensates, Eqs. (1) to (6)) have to be taken into account in a self-consistent calculation and we derived a set of six coupled gap equations for these expectation values.

The actual importance of the various condensates depends of course on the interaction. Since at present not very much is known about the effective interaction which describes the deconfined phase at moderate densities, this leaves room for surprises and possible new phenomena. For instance, in the numerical example discussed in Sec. IV A the chiral phase

transition at finite μ and zero temperature was of first order, in agreement with the general expectation. However, in Sec. IV B, where a Lagrangian with strong repulsive interactions was chosen, we found a smooth crossover. Similarly, the effect of the Lorentz non-invariant diquark condensate δ_2, which is usually neglected, was indeed found to be small in Sec. IV A, but relatively large in Sec. IV B. There it caused a reduction of the effective gap parameter $|\Delta_{eff}|$, (the minimum energy of a quasiquark excitation) by almost 50%. In general, the numerical values of the condensates depend also on how the divergent integrals are handled. In this article we used a sharp cutoff. A comparison with other approaches is yet to be done.

We also discussed the possibility of "gapless color superconductors", i.e., states with non-vanishing diquark condensates, but $|\Delta_{eff}| = 0$. We have seen that such states can in principle exist, provided the gaps of the various condensates are related to each other in a certain way (see Eq. (28)). In practice, however, all gapless superconducting states we constructed turned out to be unstable solutions of the gap equations. A similar observation was made in Ref. [15] for gapless states in the color-flavor locked phase. This suggests that gapless color superconductors might be in general unstable.

As a consequence of the spontaneous color $SU(3)$ breaking through the diquark condensates, there are also $SU(3)$ non-invariant quark-antiquark condensates in a color superconductor. For instance, the familiar scalar condensate $\langle\bar{\psi}\psi\rangle$ is in general different for quarks which participate in the diquark condensate ("red" and "green") and those which do not ("blue"). This also leads to different constituent quark masses for gapped and ungapped quarks. However, at least in our numerical examples this difference turned out to be quite small. On the other hand we found that (for equal chemical potentials) the density of the gapped quarks was considerably larger than the density of the ungapped quarks. Since the total number of red, green and blue quarks should be equal in a finite system, this could lead to the emergence of domains in which color $SU(3)$ is broken into different directions.

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FIGURES

FIG. 1. Various quantities obtained with the Lorentz invariant interaction $g_E = g_M = 2.75/\Lambda^2$ as functions of the quark chemical potential μ. Left panel: M_r (solid), Δ_1 (dashed), μ_r (dashed-dotted), μ_b (dotted). Right panel: $M_r - M_b$ (solid), $-\Delta_2$ (dashed).

FIG. 2. Left: Number densities of red quarks (solid) and blue quarks (dashed) as functions of the quark chemical potential μ. Right: Energy per quark as function of the total quark number density for a color superconducting system with equal densities of gapped and ungapped colors (dashed) and with unequal densities as given in the left panel (solid).

FIG. 3. Various quantities obtained with the Lorentz non-invariant interaction, $g_E = -1.75/\Lambda^2$ and $g_M = 4.25/\Lambda^2$. Left panel: M_r (solid), Δ_1 (dashed), $-\Delta_2$ dotted, and μ_r (dashed-dotted) as functions of the quark chemical potential μ. Right panel: Dispersion law $E_-(\vec{p})$ at $\mu = 450$ MeV. The dashed line was calculated neglecting Δ_2 in the gap equations, whereas the solid line corresponds to the exact solution.

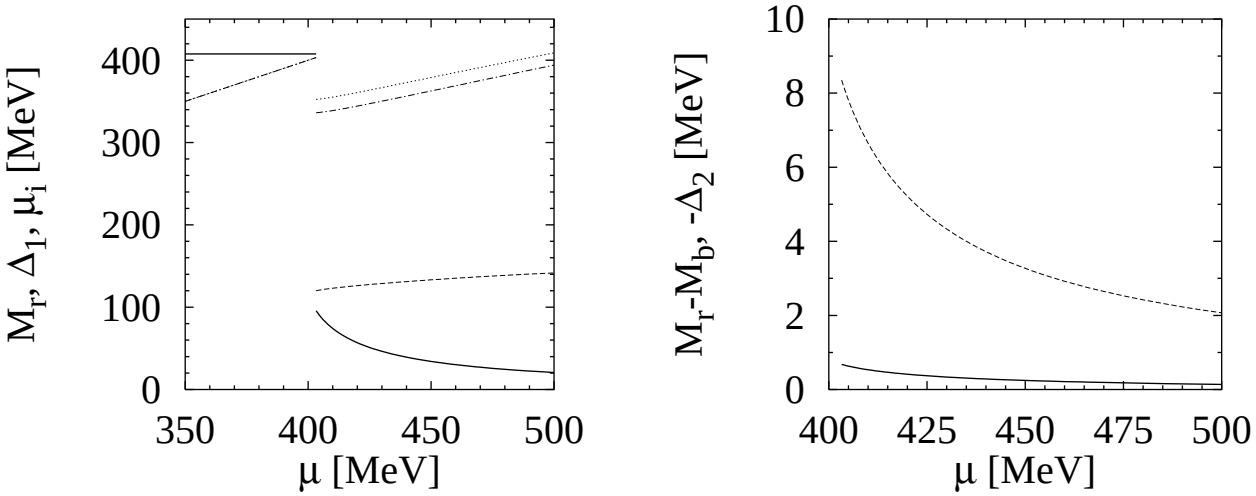
M_r, Δ_1, μ_i [MeV]
400
300
200
100
0
350
400
450
500
μ [MeV]
M_r-M_b, -Δ_2 [MeV]
10
8
6
4
2
0
400
425
450
475
500
μ [MeV]

Figure 1

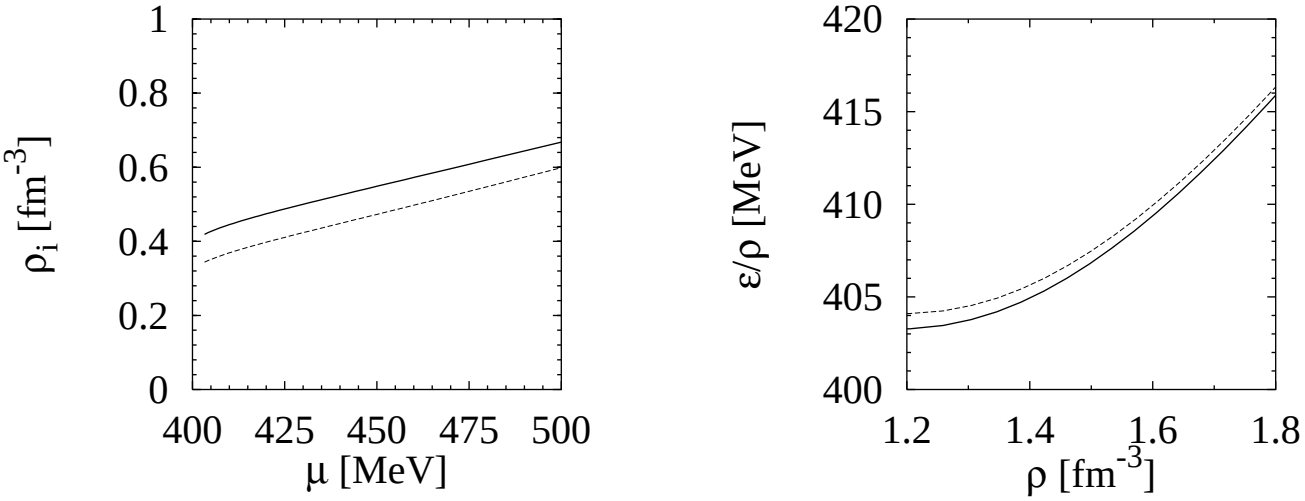
1
0.8
0.6
0.4
0.2
0
ρ_i [fm^{-3}]
400 425 450 475 500
μ [MeV]
420
415
410
405
400
ε/ρ [MeV]
1.2 1.4 1.6 1.8
ρ [fm^{-3}]

Figure 2

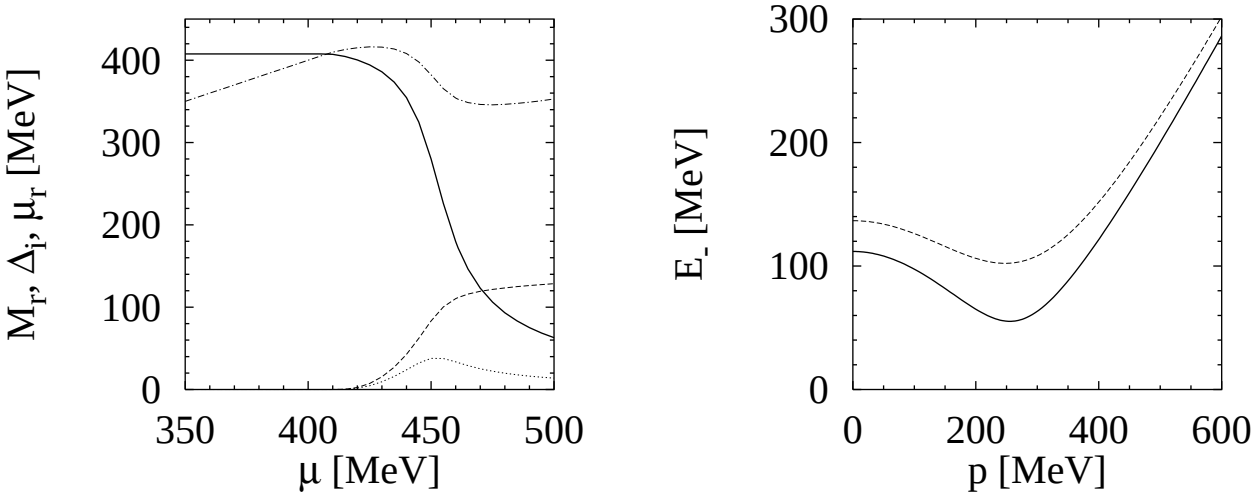
M_r, Δ_i, μ_r [MeV]
400
300
200
100
0
350
400
450
500
μ [MeV]
E_- [MeV]
300
200
100
0
0
200
400
600
p [MeV]

Figure 3