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# Duality in semileptonic inclusive $B$ -decays in potential models: regular versus singular potentials

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## Abstract

Making use of the nonrelativistic potential model for the description of mesons, and working in the Shifman–Voloshin limit, we compare the integrated rate  $\Gamma(B \rightarrow X_c l \nu)$  calculated as a sum of the individual decay rates to the quantum-mechanical analog of the OPE. In the case of a potential regular at the origin, we find a well-defined duality violation, which is, however, exponentially small. It corresponds to the charm resonances kinematically forbidden in the decay process, but apparently picked up by the OPE. For singular potentials, we do not obtain a full OPE series, but only a limited Taylor expansion, since the coefficients become infinite beyond some order. In this case, we do not find an indication of duality violation: the difference is smaller than the last term of the limited expansion. This emphasizes that the case of singular potentials, which may be relevant for QCD, deserves further study. © 2001 Published by Elsevier Science B.V. Open access under [CC BY license](#).

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The theoretical framework based on the Operator Product Expansion (OPE) determines in QCD the heavy meson inclusive decay rate as series in inverse powers of the heavy quark mass, with the coefficients proportional to the meson matrix elements of the local operators of increasing dimensions [1,2]. The calculation is based on representing the decay rate as the contour integral in the complex  $q_0$ -plane. The OPE makes the contour integrals easily calculable term by term and provides the decay rate as a  $1/m_Q$  series.

There are, however, potentially dangerous points in this calculation:

- (i) the OPE series is at best asymptotically convergent even for large absolute values of the complex  $q_0$ ;
- (ii) the integration contour for the decay rate contains a segment near the physical region, where the OPE cannot be justified [1].

This might lead to the violation of duality for the decay rate, i.e., to the difference between the OPE-calculated decay rate and the result of summing the individual decay rates of the opened channels. This issue was also discussed by N. Isgur [3].

In this Letter we discuss the semileptonic decay rate in the small velocity limit and use the nonrelativistic potential model for the description of mesons. We perform a short-time expansion in operators of increasing dimensions which we call OPE and which has indeed some common features (but also important dif-

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ferences) with the OPE expansion in the field theory. We consider the two cases: regular confining potentials<sup>1</sup> and singular potentials. Note that, in the past, the nonrelativistic potential model has been widely used in the discussion of QCD sum rules [4,5]. The question of the distinction between regular (harmonic) and singular (Coulomb) potentials has been emphasised by Novikov et al. [4] (in the context of heavy quark QCD sum rules).

We work in the Shifman–Voloshin (SV) limit  $\Lambda \ll \delta m = m_b - m_c \ll m_c, m_b$ . In this limit both the amplitude and the decay rate can be formally obtained as a *double expansion* in  $1/m_c$  and  $1/\delta m$ . We consider lowest orders in  $1/m_c$ , up to  $1/m_c^2$ , and *all orders* in  $1/\delta m$ . Note that this involves terms of much higher order than usually done when one expands in  $1/m_Q$  with  $m_b/m_c$ , or as well  $\delta m/m_Q$ , fixed. Our double expansion allows on the contrary to go much further in  $1/\delta m$ , and this might allow to display subtle duality violations.

For the regular potential we obtain the full  $1/\delta m$  expansion, which is only asymptotic to the physical width expanded to the same order in  $1/m_c$ . The difference<sup>2</sup> is of order  $\delta m/m_c^2 \exp(-\delta m/\Lambda)$ , which means exponentially small duality violation.

For the singular potential we do not obtain the full  $1/\delta m$  expansion: following the same procedure as for the regular potential leads to infinite coefficients beyond some order in  $1/\delta m$ . In this case, we find that the truncated expansion satisfies duality up to this order.

We consider the inclusive semileptonic decay  $B \rightarrow X_c l \nu$  in the SV limit and treat mesons as nonrelativistic bound states of spinless quarks in a confining potential (a detailed calculation is given in [6]). This model maximally simplifies both constructing the OPE series and calculating the sum of the exclusive channels. For the sake of argument we consider the case of leptons coupled to hadrons through the scalar current. In this case the leptonic tensor is reduced to a scalar function  $L(q^2)$ . The amplitude  $T$  depends on the two variables, and we choose them as  $q_0$  and  $\vec{q}^2$  in

<sup>1</sup> A regular potential is a potential which is an analytic function of  $\vec{r}$  at  $r = 0$ . For example, the potential  $V(r) \simeq |\vec{r}|$  falls out of this class.

<sup>2</sup> As we shall see this expansion contains only a finite number of nonzero terms.

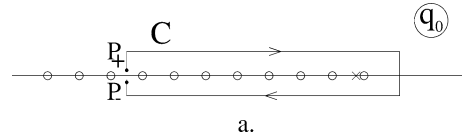


Fig. 1. Singularities of the amplitude  $T(q_0, \vec{q}^2)$  in the complex  $q_0$ -plane. Circles are poles, corresponding to low-lying charm states, and the cross marks the location of the pole in the free  $b \rightarrow c$  quark transition.

the  $B$ -rest frame:

$$T(q_0, \vec{q}^2) = \frac{1}{i} \int dx e^{-iqx} \langle B | T(J(x), J^+(0)) | B \rangle$$

$$= \sum_X \frac{| \langle B | J | X(-\vec{q}) \rangle |^2}{M_B - E_X(-\vec{q}) - q_0}. \quad (1)$$

The sum in (1) runs over all hadron states with the appropriate quantum numbers. The states are normalized as follows  $\langle \vec{p} | \vec{p}' \rangle = (2\pi)^3 \delta(\vec{p} - \vec{p}')$ , and  $E_X(-\vec{q})$  is the energy of the state  $X$  with the total 3-momentum  $-\vec{q}$ .

At fixed  $\vec{q}^2$ ,  $T(q_0, \vec{q}^2)$  has a cut in the complex  $q_0$ -plane along the real axis for  $q_0 < M_B - M_D - \vec{q}^2/2(m_c + m_d)$ , see Fig. 1.

A part of this cut for  $|\vec{q}| < q_0 < M_B - M_D - \vec{q}^2/2(m_c + m_d)$  corresponds to the decay process. The decay rate can be represented as the contour integral in the complex  $q_0$ -plane over the contour  $C(\vec{q}^2)$  (Fig. 1)

$$\Gamma(B \rightarrow X_c l \nu) = \int_{C(\vec{q}^2)} d\vec{q}^2 |\vec{q}| \int \frac{dq_0}{2\pi i} L(q^2) T(q_0, \vec{q}^2). \quad (2)$$

The contour  $C(\vec{q}^2)$  selects at any given  $\vec{q}^2$  only states kinematically allowed in the decay  $B \rightarrow X_c l \nu$ . It is tightly attached to the points  $P_{\pm}$  with the coordinates  $(|\vec{q}|, \pm i0)$ , otherwise it can be freely deformed in the region where the function  $T_0(q_0, \vec{q}^2)$  is analytic.

The amplitude can be expanded in a series

$$T(q_0, \vec{q}^2) = \sum_i c_i(q_0, \vec{q}^2) \langle B | \hat{\mathcal{O}}_i | B \rangle, \quad (3)$$

where  $\hat{\mathcal{O}}_i$  are operators of increasing dimensions and  $c_i(q_0, \vec{q}^2)$  are the  $c$ -number coefficients. Introducing the expansion (3) into (2) gives the integrated rate as an OPE series.<sup>3</sup>

<sup>3</sup> The OPE series in the potential model has an important distinction from the Wilsonian scheme in the field theory where

## 1. The model

Let us proceed along the lines of Ref. [6]. We treat the leptonic part relativistically, but for the description of mesons as bound states of spinless quarks use the nonrelativistic potential model with a confining potential. We consider the decay in the  $B$ -rest frame. The Hamiltonian of the  $b\bar{q}$  system at rest has the form

$$\begin{aligned}\widehat{H}_{bd} &= m_b + m_d + \widehat{h}_{bd}, \\ \widehat{h}_{bd} &= \vec{k}^2/2m_b + \vec{k}^2/2m_d + V_{bd}(r),\end{aligned}$$

such that

$$\begin{aligned}(\widehat{h}_{bd} - \epsilon_B)|B\rangle &= 0, \\ (\widehat{H}_{bd} - M_B)|B\rangle &= 0, \\ M_B &= m_b + m_d + \epsilon_B.\end{aligned}$$

The Hamiltonian of the  $c\bar{q}$  system produced in the semileptonic  $b \rightarrow c\ell\nu$  decay reads

$$\begin{aligned}\widehat{H}_{cd}(\vec{q}) &= m_c + m_d + (\vec{k} + \vec{q})^2/2m_c \\ &\quad + \vec{k}^2/2m_d + V_{cd}(r).\end{aligned}$$

The eigenstates of this Hamiltonian are  $|D_n(\vec{q})\rangle$  such that

$$(\widehat{H}_{cd}(\vec{q}) - E_{D_n}(\vec{q}))|D_n(\vec{q})\rangle = 0,$$

where

$$\begin{aligned}E_{D_n}(\vec{q}) &= M_{D_n} + \vec{q}^2/2(m_c + m_d), \\ M_{D_n} &= m_c + m_d + \epsilon_{D_n}.\end{aligned}$$

The  $Q\bar{q}$  potential can be expanded as follows:

$$V_{Qq} = V_0 + V_1/2m_Q + V_2/2m_Q^2 + \dots$$

contributions of distances below the scale  $1/\mu$  are referred to the Wilson coefficients, while contributions of distances above this scale are referred to the matrix elements. As a result, both the Wilson coefficients and the matrix elements acquire the  $\mu$ -dependence. In the potential model, the  $T$ -product of the two currents is also expanded in a series of operators of increasing dimensions, but the resulting  $c$ -number coefficients and the matrix elements in Eq. (3) are scale independent.

## 2. Sum rules

The relationship between the sum over the individual channels and the meson matrix elements of the operators is established by the sum rules. Let us introduce  $\delta_n(\vec{q})$  through the relation  $(\delta_n(\vec{q}) = \epsilon_{D_n} - \epsilon_B - \vec{q}^2 m_d/2m_c(m_c + m_d))$

$$M_B - q_0 - E_{D_n}(\vec{q}) = \delta m - q_0 - \vec{q}^2/2m_c - \delta_n(\vec{q}). \quad (4)$$

The  $\delta_n(\vec{q})$  is the eigenvalue of the operator  $\delta H(\vec{q})$

$$M_B - \widehat{H}_{cd}(\vec{q}) = \delta m - \vec{q}^2/2m_c - \delta H(\vec{q}) \quad (5)$$

with  $|D_n(\vec{q})\rangle$  the corresponding eigenstates. The sum rules are obtained by inserting the full system of the eigenstates  $|D_n(\vec{q})\rangle$  into  $\langle B|(\delta H(\vec{q}))^i|B\rangle$ :

$$\langle B|(\delta H(\vec{q}))^i|B\rangle = \sum_{n=0}^{\infty} |F_n(\vec{q})|^2 (\delta_n(\vec{q}))^i, \quad (6)$$

where  $F_n(\vec{q}) = \langle B|D_n(\vec{q})\rangle$  is the  $B \rightarrow D_n$  transition form factor. This relation represents the sum over all  $c\bar{d}$  resonances in terms of the  $B$ -meson matrix element of the operators  $(\delta H(\vec{q}))^i$ . For the potential regular at the origin  $r = 0$  the sum over  $n$  is convergent for any  $i$ , whereas for the singular potential both sides of Eq. (6) are convergent for small  $i$  and diverge for large  $i$ . At the moment we proceed formally and discuss this problem in more detail in Section 5.

## 3. Duality relation for the amplitude

Making use of the sum rules (6), we represent the amplitude as a sum of the operators:

$$\begin{aligned}T(q_0, \vec{q}) &= \sum_{n=0}^{\infty} \frac{|F_n(\vec{q})|^2}{M_B - q_0 - E_n(\vec{q})} \quad (7)\end{aligned}$$

$$= \frac{1}{\delta m - \frac{\vec{q}^2}{2m_c} - q_0} \sum_{n=0} \sum_{i=0} \frac{|F_n(\vec{q})|^2 (\delta_n(\vec{q}))^i}{(\delta m - \frac{\vec{q}^2}{2m_c} - q_0)^i} \quad (8)$$

$$= \frac{1}{\delta m - \frac{\vec{q}^2}{2m_c} - q_0} \sum_{i=0} \frac{\langle B|(\delta H(\vec{q}))^i|B\rangle}{(\delta m - \frac{\vec{q}^2}{2m_c} - q_0)^i}. \quad (9)$$

This expression is the duality relation for the amplitude: the sum (7) runs over the infinite number of the

charm resonances, and the sum (8) runs over the infinite number of the operators of the increasing dimensions (the OPE series). In fact, the location of singularities in the complex  $q_0$ -plane in the series (7) and (9) is quite different: in (7) it is an infinite set of single poles at the different locations corresponding to different charm resonances, and in (8) it is an infinite set of poles of the increasing order at the same point.

However, this set of equations is only a formal one; in fact, (7) is a summable series leading to a finite result in all cases; on the other hand, the situation of Eq. (9) is more subtle. In the singular case, the coefficients are infinite beyond some order, and one must accordingly truncate the series. In the regular case, the Eq. (9) is only an asymptotic series: notice that the geometric sum over  $i$  in Eq. (8) has a domain of convergence which is repelled to infinity with  $n$ .

Let us illustrate it with a simple example: assume that  $F_n^2 \simeq e^{-n}$  and  $E_n \simeq n$ . Then the analog of the above equations takes the form

$$\sum_{n=0}^{\infty} \frac{e^{-n}}{z-n} = \sum_{n=0}^{\infty} \frac{e^{-n}}{z} \sum_{i=0}^{\infty} \binom{n}{z}^i \simeq \frac{1}{z} \sum_{i=0}^{\infty} \frac{i!}{z^i}. \quad (10)$$

The last step is obtained by changing the order of summation and using the relation  $\sum_{n=0}^{\infty} e^{-n} n^i \simeq i!$ . The series (10) in  $i$  is only asymptotic and not even Borel summable.

Such a factorial divergence appears in the example of M.A. Shifman [7]. In this example, the residues are constant with the excitation number. This would imply that the direct method of our Letter will give formally infinite coefficients for the expansion, which means that it fails completely. In our example, on the contrary, the residues are rapidly decreasing, and all the coefficients are finite. The interest of our example is to show that nevertheless a factorial divergence is also appearing.

From the amplitude  $T$  under the form Eq. (7) or Eq. (9), respectively, by integration over the same contour  $C$ , we can obtain either the width as a sum over the exclusive final states, or as the OPE series. The expression (9) is an accurate approximation to (7) only when  $q_0$  is far from the singularities of  $T(q_0, \vec{q})$ . The contour  $C$  can be deformed away from the singularities except near its fixed end points. When integrating over  $q_0$  this is a possible source of discrepancy, i.e., of duality violation. Consequently,

we are now going to estimate the integral of expression (7), i.e. the sum over the exclusive channels, and the integral of expression (9), i.e., the OPE prediction, and compare both results.

#### 4. The OPE calculation of the decay rate

Let us first proceed with the amplitude in the form (9) and obtain the OPE expression for the decay rate. We consider the leptonic tensor of the general form  $L(q^2) = (q^2)^N$ . For technical reasons, it is convenient to isolate  $h_{bd}$  in the expression for  $\delta H(\vec{q})$  as follows

$$\delta H(\vec{q}) = h_{bd} - \epsilon_B + \frac{\vec{k}\vec{q}}{m_c} + \left( \frac{1}{m_c} - \frac{1}{m_b} \right) \frac{\vec{k}^2 + V_1}{2}. \quad (11)$$

Substituting (11) in (9) and performing the necessary integrations gives a series in  $1/m_c$  [6]

$$\begin{aligned} & \frac{\Gamma^{\text{OPE}}(B \rightarrow X_c l \nu)}{\Gamma(b \rightarrow c l \nu)} \\ &= 1 + \frac{\langle B | \vec{k}^2 | B \rangle}{2m_c^2} - (2N+3) \frac{\langle B | V_1 | B \rangle}{2m_c^2} \\ &+ \sum_{i=1}^{2N+3} (-1)^i \frac{C_{2N+5}^{i+2}}{2N+5} \frac{\langle B | \widehat{O}_i | B \rangle}{m_c^2 \delta m^i} + O\left(\frac{\Lambda^2 \delta m}{m_c^3}\right), \end{aligned} \quad (12)$$

with  $C_n^i = \frac{n!}{i!(n-i)!}$  and  $\widehat{O}_i = \vec{k}(h_{bd} - \epsilon_B)^i \vec{k}$ . An important feature of the OPE series (12) is that the leading-order term reproduces the free-quark decay rate, and the first correction emerges only in the  $1/m_c^2$  order (cf. [1,2]).

#### 5. Summation of the exclusive channels

Now let us sum the rates of the exclusive channels. The  $B \rightarrow D_n$  transition form factors have the form [6]

$$\begin{aligned} F_0^2(\vec{q}) &= 1 - \rho_0^2 \vec{q}^2 / m_c^2 + O(\vec{q}^4 / m_c^4) \\ &+ O(\delta m^2 \beta^2 / m_c^4), \\ F_n^2(\vec{q}) &= \rho_n^2 \vec{q}^2 / m_c^2 + O(\vec{q}^4 / m_c^4) + O(\delta m^2 \beta^2 / m_c^4). \end{aligned}$$

Since  $|\vec{q}| \leq \delta m$  in the decay region, these expressions allow calculating the decay rate to the accuracy

$\delta m^2/m_c^2$ . Explicitly, we obtain [6]:

$$\begin{aligned} & \frac{\Gamma(B \rightarrow D_0 l \nu)}{\Gamma(b \rightarrow c l \nu)} \\ &= 1 - \frac{3\rho_0^2}{2N+5} \frac{\delta m^2}{m_c^2} + \frac{3}{2} \frac{m_d}{1+m_d/m_c} \frac{\delta m}{m_c^2} \\ & \quad - (2N+3) \frac{\langle B | \vec{k}^2 + V_1 | B \rangle}{2m_c^2}, \\ & \frac{\Gamma(B \rightarrow D_n l \nu)}{\Gamma(b \rightarrow c l \nu)} \\ &= \frac{3\rho_n^2}{2N+5} \frac{\delta m^2}{m_c^2} - 3(\rho_n^2 \Delta_n) \frac{\delta m}{m_c^2} \\ & \quad + \frac{1}{m_c^2} \sum_{i=2}^{2N+5} \frac{(-1)^i C_{2N+5}^i}{2N+5} \frac{(3\rho_n^2 \Delta_n^i)}{\delta m^{i-2}}, \end{aligned} \quad (13)$$

where  $\Delta_n = \epsilon_{D_n} - \epsilon_{D_0}$ .

The main contribution is given by the  $B \rightarrow D_0$  transition. Excited states contribute only starting from the  $(\delta m)^2/m_c^2$  order in the SV limit. Notice that *each of the exclusive rates contains terms of the order  $\delta m^2/m_c^2$  and  $\Lambda \delta m/m_c^2$  which are absent in the OPE series.*

Summing over all opened exclusive channels gives

$$\begin{aligned} & \frac{\Gamma(B \rightarrow X_c l \nu)}{\Gamma(b \rightarrow c l \nu)} \\ &= 1 - \frac{\delta m^2}{m_c^2} \frac{3}{2N+5} \left( \rho_0^2 - \sum_{n=1}^{n_{\max}} \rho_n^2 \right) \\ & \quad + 3 \frac{\delta m}{m_c^2} \left( \frac{m_d/2}{1+m_d/m_c} - \sum_{n=1}^{n_{\max}} \rho_n^2 \Delta_n \right) - (2N+3) \\ & \quad \times \frac{\langle B | \vec{k}^2 + V_1 | B \rangle}{2m_c^2} \\ & \quad + \sum_{i=2}^{2N+5} \frac{(-1)^i C_{2N+5}^i}{2N+5} \frac{3(\sum_{n=1}^{n_{\max}} \rho_n^2 \Delta_n^i)}{m_c^2 \delta m^{i-2}}. \end{aligned} \quad (14)$$

The sum over the charm resonances is truncated at  $n_{\max}$ , which is the total number of the resonance levels opened at  $q^2 = 0$ . For the confining potential and in the SV limit  $n_{\max}$  is found from the relation  $\Delta_{n_{\max}} \simeq \delta m$ .

## 6. Check of duality for regular potentials

The transition radii in the expression (14) are not independent and related to each other through the sum rules. These sum rules can be obtained from (6). Expanding both sides of (6) in powers of  $1/m_Q$  and taking the linear  $\vec{q}^2$  term gives the set of the sum rules [6]: for  $i = 0$  one finds the Bjorken sum rule [8], for  $i = 1$  — the Voloshin sum rule [9], for  $i \geq 2$  — higher moment sum rules:

$$\begin{aligned} i = 0: & \quad \sum_{n=1}^{\infty} \rho_n^2 = \rho_0^2, \\ i = 1: & \quad \sum_{n=1}^{\infty} \rho_n^2 \Delta_n = \frac{m_d/2}{1+m_d/m_c}, \\ i \geq 2: & \quad \sum_{n=1}^{\infty} \rho_n^2 \Delta_n^i = \frac{1}{3} \langle B | \vec{k} (h_{bd} - \epsilon_B)^{i-2} \vec{k} | B \rangle. \end{aligned} \quad (15)$$

Using these relations to rewrite the OPE result (12) as the sum over hadronic resonances, the difference between the OPE and the exclusive sum (the duality-violating contribution) explicitly reads

$$\begin{aligned} \delta_\Gamma &\equiv \frac{\Gamma^{\text{OPE}}(B \rightarrow X_c l \nu) - \Gamma(B \rightarrow X_c l \nu)}{\Gamma(b \rightarrow c l \nu)} \\ &= 3 \frac{\delta m^2}{m_c^2} \sum_{i=0}^{2N+5} \frac{(-1)^i C_{2N+5}^i}{(2N+5) \delta m^i} \sum_{n>n_{\max}} \rho_n^2 (\Delta_n)^i \\ & \quad + O\left(\frac{\Lambda^2 \delta m}{m_c^3}\right) \\ &= \frac{\delta m^2}{m_c^2} \frac{3}{2N+5} \sum_{n>n_{\max}} \rho_n^2 \left(1 - \frac{\Delta_n}{\delta m}\right)^{2N+5} \\ & \quad + O\left(\frac{\Lambda^2 \delta m}{m_c^3}\right). \end{aligned} \quad (16)$$

Quite remarkably,  $\delta_\Gamma$  happens to be equal to the sum of the extrapolated widths for charm states beyond the kinematical limit. A similar expression is found in QCD<sub>2</sub> [12]. Clearly, the duality-violating effect is connected with the charm states forbidden kinematically in the decay process. Notice that  $\Gamma^{\text{OPE}}(B \rightarrow X_c l \nu) - \Gamma(B \rightarrow X_c l \nu) < 0$ , because  $\Delta_n > \delta m$  for  $n > n_{\max}$ , and  $2N+5$  is odd.

To estimate the size of the duality-violation effects, the behavior of the *transition radii* and the relation between  $\Delta_n$  and  $n_{\max}$ , which will be given by the

behaviour of the *excitation energies* also at large  $n$ , are needed. For quite a general form of the confining potential we can write the following relations for  $\Delta_n$  at large  $n$   $\Delta_n \geq \Lambda C n^a$  for  $n > n_{\max}$  and  $\Delta_{n_{\max}} = \Lambda C (n_{\max})^a \simeq \delta m$ , with  $C$  and  $a$  some positive numbers. In particular, this estimate is valid for the confining potentials with a power behavior at large  $r$ . This estimate for  $\Delta_n$  is only depending on the behaviour of the potential at large distances.

The behavior of the radii  $\rho_n^2$  at large  $n$  are then connected with the finiteness of the r.h.s. of the sum rules (15): for a potential regular at  $r = 0$ , the matrix elements in the r.h.s. of the sum rules are finite for any  $i$ , which means that the radii  $\rho_n^2$  are decreasing with  $n$  faster than any power. Essentially this means that  $\rho_n^2 \simeq \exp(-n)$ , and, therefore, the duality-violating effect in the decay rate in (16) is of order  $\delta_\Gamma \simeq \delta m^2/m_c^2 \exp(-\delta m/\Lambda)$ . One of such examples, the harmonic oscillator potential, is discussed in [10].

## 7. Singular potentials

However, if the potential is singular at  $r = 0$ , the situation changes dramatically. First, only a few first number of the matrix elements  $\langle B|\widehat{O}_i|B\rangle$  are finite.<sup>4</sup>

We can try to proceed along the same lines but then have to truncate the series in  $1/\delta m$  at the last finite term. We want to estimate the difference between this truncated series and the exclusive sum.

Let us illustrate this considering a potential with a Coulomb behavior at small  $r$ ,  $V \simeq -\alpha/r$ , and confining at large  $r$ . Then  $\langle B|\vec{k}(h_{bd} - \epsilon_B)^i \vec{k}|B\rangle$  are finite for  $i \leq 1$ , but diverge starting from  $i = 2$ . We then find that

<sup>4</sup> The appearance of infinite coefficients in the OPE series is probably due to a breakdown of the power series expansion, for instance by fractional powers or logarithms of  $m_Q$  as seems to be the case in the pure Coulomb case [11]. For similar phenomena in a perturbation expansion, see [13].

In particular, it has been noted by Novikov et al. [4] that, in the case of the Coulomb potential, half integral powers of the expansion parameter occur in the calculation of the moments of the ratio  $R$ . However, in our case, preliminary work shows that there are not only half integer powers of  $\Lambda/\delta m$ , but also logarithms of this quantity. In fact, we note in this respect that the full Schwinger formula for the ratio  $R$  contains also logarithms in addition to half integer powers [14].

$$\rho_n^2 \leq \frac{1}{n^{1+\varepsilon}} \left( \frac{1}{n^a} \right)^3. \quad (17)$$

Such  $\rho_n^2$  lead to the estimate

$$\delta_\Gamma \simeq \frac{\Lambda^2}{m_c^2} \left( \frac{\Lambda}{\delta m} \right)^{(1+\varepsilon/a)}. \quad (18)$$

More generally, if the above matrix element begins to diverge for some value  $i = K + 1$ , the formulas are to be replaced by:

$$\rho_n^2 \leq \frac{1}{n^{1+\varepsilon}} \left( \frac{1}{n^a} \right)^{K+2}, \quad (19)$$

$$\delta_\Gamma \simeq \frac{\Lambda^2}{m_c^2} \left( \frac{\Lambda}{\delta m} \right)^{(K+\varepsilon/a)}. \quad (20)$$

Notice that this  $\delta_\Gamma$  is smaller than the last retained term in the OPE series which is of order  $\frac{\Lambda^2}{m_c^2} \left( \frac{\Lambda}{\delta m} \right)^K$ . Therefore, the ‘duality violation’ is just smaller than the last retained term as for the asymptotic series. This means in fact that there is *no indication of duality violation at this computable order*. This is independent of  $a$ , therefore of the *large distance behavior of the potential*.

## 8. Conclusion

Summarizing our results, the amplitude  $T(q_0, \vec{q}^2)$  (the  $T$ -product, Eq. (1)) can be expanded in inverse powers of  $\delta m - \vec{q}^2/2m_c - q_0$ , the so-called OPE expansion. Exact duality would mean that the OPE series was convergent and equal to  $T$ . Actually, this is not exactly the case. Even in the favourable case of the *regular potentials* (at  $\vec{r} = 0$ ), the OPE series is not convergent, it is only asymptotic to the actual  $T$ . For *singular potentials*, the coefficients are simply infinite beyond a certain order.

Besides these problems concerning the amplitude, additional problems appear for the expansion of the width, which is given by a contour integral of  $T$  in the  $q_0$  complex plane: the OPE expansion is accurate far from the singularities in  $q_0$ , while the contour has fixed end points in the complex plane close to the singularities (Fig. 1). In view of this situation, we have computed explicitly the difference between the OPE and the actual width. For singular potentials, the series

must be truncated, and the difference is found smaller than the last retained term.

As to the perspectives opened by this work, we must first emphasize that singular potentials seem more interesting than regular ones. Indeed, in QCD the effective quark potential is singular, a smoothed Coulomb singularity. Moreover, in  $\text{QCD}_2$ , one can suspect some similarity with a linear potential  $|\vec{r}|$ , which is also singular at the origin in the sense of this Letter. For a singular potential, we have seen that the entire series must be truncated at some order, because the coefficients become eventually infinite. We think that such infinite coefficients in an entire series expansion correspond to the fact that the correct expansion is not entire but must include fractional powers and/or logarithms in the expansion parameter, i.e.,  $\delta m$ . In QCD, one can argue that the operator matrix elements are finite due to renormalisation, but nevertheless the coefficients still contain logarithms of heavy masses. In the nonrelativistic case, the object of the present Letter, the method which has been followed does not lead to definite conclusions as regards duality for singular potentials: namely, to the order we are able to calculate in this Letter, we find that there is no duality violation, but this leaves open the question of duality violation at some higher order.<sup>5</sup> To proceed further, one would

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<sup>5</sup> In the context of  $\text{QCD}_2$ , one has demonstrated duality up to the order  $1/m_Q^4$  and it may be believed that duality has been fully demonstrated in higher orders [12]. However, a comment is in order here. In [12], it was shown that the matrix element of the leading operator  $\langle B|\bar{Q}Q|B\rangle$  is dual to the sum of the widths of the full tower of resonances. Therefore, one can suspect that there is a difference between the actual width and the OPE, that is of higher order  $1/m_Q^5$ , corresponding to the extrapolated width of the kinematically forbidden states. This difference, however, has the same order  $1/m_Q^5$  as the matrix elements of the higher dimension operators [12]. It was then *assumed* that both quantities are dual to each other, but the corresponding OPE coefficients were not calculated and we have not found where this assumption was demonstrated.

have to devise new methods to obtain the above conjectured generalized expansions.

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