Bias and the Power Spectrum beyond the Turnover

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ABSTRACT

Threshold biasing of a Gaussian random field gives a linear amplification of the reduced two point correlation function at large distances. We show that for standard cosmological models this *does not* translate into a linear amplification of the power spectrum (PS) neither at small k not at large k. For standard CDM type models the "turn-over" at small k of the original PS disappears in the PS of the biased field for the physically relevant range of threshold parameters ν. In real space this difference is manifest in the asymptotic behaviour of the normalised mass variance in spheres of radius R, which changes from the "super-homogeneous" behaviour $\sigma^2(R) \sim R^{-4}$ to a Poisson-like behaviour $\sigma_\nu^2(R) \sim R^{-3}$. This qualitative change results from the intrinsic stochasticity of the threshold sampling. While our quantitative results are specific to the simplest threshold biasing model, we argue that our qualitative conclusions should be valid generically for any biasing mechanism involving a scale-dependent amplification of the correlation function. One implication is that the real-space correlation function will be a better instrument to probe for the underlying Harrison Zeldovich spectrum in the distribution of visible matter, as the characteristic asymptotic negative power-law $\xi(r) \sim -r^{-4}$ tail is undistorted by biasing.

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The concept of bias has been introduced by Kaiser (1984), primarily to explain the observed difference in amplitude between the correlation function of galaxies and that of galaxy clusters. In this context the underlying distribution of dark matter is treated as a correlated Gaussian density field. The galaxies of different luminosities or galaxy clusters, are interpreted as the peaks of the matter distribution, which have collapsed by gravitational clustering. Different kind of objects are selected as peaks above a given threshold, with a change in the threshold selecting different regions of the underlying Gaussian field, corresponding to fluctuations of differing amplitudes. The reduced two-point correlation function of the selected objects is then that of the peaks $\xi_\nu(r)$, which is enhanced with respect to that of the underlying density field $\xi(r)$. In a previous paper (Gabrielli, Sylos Labini & Durrer, 2000 - GSLD00) some of us have discussed the problematic aspects of this mechanism. In particular the amplification of the correlation function is in fact only linear in the regime in which $\xi_\nu(r) \ll 1$ (see also Politzer & Wise, 1984). In the region of most observational relevance (where $\xi_\nu(r) \gg 1$) the correlation function is actually distorted at least exponentially. Furthermore we have drawn attention to the fact that the amplification of the correlation function by biasing reflects simply that the distribution of peaks is *more clustered* because peaks are exponentially *sparser*.

In this letter we discuss a different aspect of this model for bias. We are interested in understanding the effect of biasing on the power spectrum (PS). In particular we address here a qualitative change that is caused to the matter perturbations in in standard cosmological models. In real space this change manifests itself in a change from sub-Poissonian behaviour of the mass variance at large scales in the underlying density field, to Poissonian behaviour of the same quantity for the biased field. In k space this implies a distortion of the PS at small k. Our analysis shows that this effect can be very important observationally, as it can make the "turn-over" in the dark matter PS disappear from the PS of visible matter. Furthermore it shows the importance of measuring not just not just the PS of visible objects, but also their real space correlation properties. Earlier works about the effect of biasing in the PS can be found e.g. Coles (1993) and Scherrer & Weinberg (1998).

Before considering threshold biasing, we recall the relevant part of the analysis given in Gabrielli, Joyce & Sylos Labini (2002) (hereafter GJSL02). In this paper we have discussed the meaning of the condition $P(0) = 0$ satisfied by the PS of all current standard type cosmological models (with Harrison-Zeldovich like spectra $P(k) \sim k$ at small k). While this point is often noted in the cosmological literature (see e.g. Padmanabhan 1993), its significance and implications are not correctly appreciated (see GJSL02 for discussion). It implies the requirement that the integral over all space of the correlation function vanishes, meaning that in the system there is an exact balance between correlations and anti-correlations at all scales. This is a highly non-trivial, non-local, condition on the distribution. Its specificity can be highlighted by the following classification of all stationary stochastic processes into three categories: (i) For $P(0) = \infty$ the fluctuations are like those in a critical long range correlated system, (ii) for $P(0) = constant > 0$ the system is Poisson-like at large scales e.g. any short-range positively correlated system such as a quasi-ideal gas at thermal equilibrium, and (iii) for $P(0) = 0$ the system is what we have termed "super-homogeneous". The reason for the use of this last term comes from the fact that the three categories are distinguished most strikingly in real space by the large distance behaviour of the mass variance in spheres, as one can show that $P(0) = \lim_{V\to\infty} \frac{\langle(\Delta M(V))^2\rangle}{\rho_o^2 V}$ where $\langle(\Delta M(V))^2\rangle$ is the mass variance in a volume V (and ρ_o the mean mass density). In the Poisson type distribution this variance is proportional to the volume of the sphere, while in the first category (critical systems) it grows more rapidly (with a limiting behaviour of the volume squared), while in the last (super-homogeneous distributions) the growth is slower than the Poissonian one. In particular the case of the H-Z spectrum marks the transition to the limiting slowest possible growth of this quantity for *any* stochastic distribution of points (Beck 1987), which is a growth proportional to the surface of the sphere.

These super-homogeneous distributions are encountered in various contexts in statistical physics. They are described in this context as glass-like: they are highly ordered distributions like a lattice, but with full statistical isotropy and homogeneity. In Gabrielli et al. (2002) an example of a system with such correlations at thermal equilibrium is given, and a modification of this same system which should give precisely the correlation of a standard cosmological model is described.

We now turn to the threshold biasing mechanism. Following Kaiser (1984) we consider a stationary, isotropic and correlated continuous Gaussian random field, $\delta(\mathbf{x})$, with zero mean and variance $\sigma^2 = \langle\delta(\mathbf{x})^2\rangle$ in a volume V as $V \to \infty$. The marginal one-point probability density function of δ is $\mathcal{P}(\delta) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\delta^2}{2\sigma^2}}$. Using $\mathcal{P}(\delta)$, we can calculate the fraction of the volume V with $\delta(\mathbf{x}) \geq \nu\sigma$, $Q_1(\nu) = \int_{\nu\sigma}^{\infty} \mathcal{P}(\delta) d\delta$. The correlation function between the values of $\delta(\mathbf{x})$ in two points separated by a distance r is given by $\xi(r) = \langle\delta(\mathbf{x})\delta(\mathbf{x} + r\mathbf{n})\rangle$. By definition, $\xi(0) = \sigma^2$. In this context, stationarity means that the variance, σ^2, and the correlation function, $\xi(r)$, do not depend on $\mathbf{x}$. Statistical isotropy means that $\xi(r)$ does not depend on the direction $\mathbf{n}$. The goal is to determine the correlation function of local maxima from the correlation function of the underlying density field. The problem can be simplified (Kaiser 1984) by computing the correlation of *regions* above a certain threshold $\nu\sigma$ instead of the correlations of *maxima*. However, these quantities are closely related for values of ν significantly larger than 1. We define the threshold density, $\theta_\nu(\mathbf{x})$ by

$$\theta_\nu(\mathbf{x}) \equiv \theta(\delta(\mathbf{x}) - \nu\sigma) = \begin{cases} 1 & \text{if} \quad \delta(\mathbf{x}) \geq \nu\sigma \\ 0 & \text{else.} \end{cases} \tag{1}$$

Note the qualitative difference between δ which is a weighted density field, and θ_ν which just defines uniform domains, all having equal weight, and $\langle\theta_\nu(\mathbf{x})\rangle = Q_1$.

Let us now consider how the biasing changes the distribution in relation to the classification we have given in terms of $P(0)$. In what follows we show, for $\nu > \nu_o > 0$, (ν_o given below)

$$P_\nu(0) > P(0) \tag{2}$$

where $P_\nu(k)$ and $P(k)$ are the PS of the biased and underlying field respectively, i.e. the Fourier transform of $\xi_\nu(r)$ and of the normalised underlying correlation function $\xi_c(r) \equiv \xi(r)/\sigma^2$ respectively. This result is independent of $\xi_c(r)$. The correlation function $\xi_\nu(r)$ of the biased field is given (Kaiser 1984) by the expression

$$Q_1(\nu)^2(\xi_\nu(r)+1) = \frac{1}{2\pi\sqrt{1-\xi_c^2(r)}} \int_\nu^\infty \int_\nu^\infty d\delta d\delta'$$
$$\times \exp\left(-\frac{(\delta^2+\delta'^2) - 2\xi_c(r)\delta\delta'}{2(1-\xi_c^2(r))}\right) \tag{3}$$

where the integrand on the right hand side is the two point joint probability density for the Gaussian field Using the expression for $Q_1(\nu)$ given above, one can recast this after a simple change of variables into the form

$$\xi_\nu(r) = \frac{\int_\nu^\infty dx e^{-x^2/2} \int_\mu^\nu dy e^{-y^2/2}}{[\int_\nu^\infty dx e^{-x^2/2}]^2}. \tag{4}$$

where $\mu = (\nu - \xi_c x)/\sqrt{1-\xi_c^2}$. In this form it is evident that $\xi_c(r) = 0 \Leftrightarrow \xi_\nu(r) = 0$ and that $sign[\xi_\nu(r)] = sign[\xi_c(r)]$.

Taylor expanding this expression about $\xi_c = 0$, we find[1]

$$\xi_\nu(r) = b_1(\nu)\xi_c(r) + b_2(\nu)\xi_c^2(r) + ... \tag{5}$$

[1]For the expansion to all orders see Jensen & Szalay (1986).

with

$$b_1(\nu) = e^{-\nu^2/2} \frac{\int_\nu^\infty dx x e^{-x^2/2}}{[\int_\nu^\infty dx e^{-x^2/2}]^2} \tag{6}$$

$$b_2(\nu) = \frac{1}{2}\nu e^{-\nu^2/2} \frac{\int_\nu^\infty dx (x^2-1) e^{-x^2/2}}{[\int_\nu^\infty dx e^{-x^2/2}]^2} . \tag{7}$$

The first term gives the linear relation obtained by Kaiser (1984), as $b_1(\nu) \approx \nu^2$ for $\nu \gg 1$, valid in the regime $|\xi_c| \ll 1$ and $|\xi_\nu| \ll 1$. It is easy to check that $b_2(\nu)$ is positive definite for $\nu \geq 0$ (and $b_2(\nu) \approx \nu^4/2$ for $\nu \gg 1$), so that to this order in $\xi_c(r)$ one has the bound

$$\xi_\nu(r) > b_1(\nu)\xi_c(r) \;\text{ for }\; \xi_\nu(r) \neq 0 \text{ and } \nu > 0 . \tag{8}$$

If $|\xi_c(r)| \ll 1$ at all r this bound suffices to give the desired result (2) for all values of ν such that $b_1(\nu) \geq 1$. As $b_1(\nu)$ is a monotonically increasing function of ν (with $b_1(0) = 2/\pi$) this is equivalent to the requirement $\nu \geq \nu_o$ with ν_o such that $b_1(\nu_o) = 1$ (i.e. $\nu_o \simeq 0.303$). To show that there is a value of ν above which Eq. (2) is indeed satisfied for all permitted values of ξ_c, it suffices to find the threshold value $\nu_1 \geq \nu_o$ such that for all $\nu \geq \nu_1$ one has

$$sign\left[\frac{d\xi_\nu(r)}{d\xi_c(r)} - b_1(\nu)\right] = sign[\xi_c] \quad \forall r . \tag{9}$$

In fact this is a sufficient condition to have the exact curve $\xi_\nu(\xi_c)$, given by Eq. (4), all above the line $\xi_\nu = \xi_c$ for all ξ_c. We have found numerically, using Eq. (4), that this condition is satisfied for $\nu_1 \simeq 0.38$.

Note that, if the condition (8) holds, this means simply that, relative to the asymptotic ($|\xi_c| \ll 1$ and $|\xi_\nu| \ll 1$) linearly biased regime in which $\xi_\nu \approx b_1(\nu)\xi_c$, the anti-correlated regions are less amplified ($|\xi_\nu| < b_1(\nu)|\xi_c|$) than the positively correlated regions ($|\xi_\nu| > b_1(\nu)|\xi_c|$). Thus the integral over the biased correlation function is always positive, and the bound (2) thus holds. Further it is easy to see that $P_\nu(0)$ is finite if $P(0)$ is: ξ_ν is bounded for any value of ν, and, has the same convergence properties as ξ_c at large distances. This implies that, if the integral of ξ_c over all space converges, then also that of ξ_ν does.

In terms of the classification of distributions by $P(0)$ we thus draw the following conclusion: Both the critical type system (with $P(0) = \infty$) and Poisson type system (with $P(0) = constant > 0$) remain in the same class; the super-homogeneous distribution (with $P(0) = 0$) however becomes Poissonian ($P_\nu(0) = constant > 0$). The essential reason for these changes is simple: as discussed above the behaviour of the PS is the same as that of the mass variance at asymptotically large scales. The biasing process is stochastic in nature, and introduces a variance in the number of objects which is proportional to the volume. This new variance will dominate asymptotically over that of the original distribution only if the latter is super-homogenous (i.e. its asymptotic normalised variance is sub-poissonian, decaying faster than Poisson). Consider for example the case of a perfect lattice, which is a super-homogeneous distribution ($P(0) = 0$) in which the normalised variance $\sigma^2(R) = \langle(\Delta M(R))^2\rangle/\langle M(R)\rangle^2$ in a sphere of radius R decays asymptotically as $1/R^4$. The distribution obtained by keeping (or rejecting) each point with probability p (or $1-p$) is described by a simple binomial distribution, with a variance $\sigma^2(R) \propto p(1-p)/N \propto 1/R^3$ (N being the mean number of points inside a sphere). Biasing is not such a purely random sampling, but the effect of stochasticity as a source of Poisson variance at large scales is similar. Translated in terms of the PS it gives the result we have derived.

Now let us turn to the implications of this result for cosmological models. Since such models have $P(0) = 0$ in the full matter spectrum, it is evident that we cannot have the behaviour $P_\nu(k) \propto b_1(\nu)P(k)$ for small k which one might naively infer from the fact that $\xi_\nu(r) \approx b_1(\nu)\xi_c(r)$ for large separations. Inevitably a non-linear distortion of the biased PS at small k relative to the underlying one is induced. How important can the effect be qualitatively for a realistic cosmological model? To answer this question we consider the simple model PS $P(k) = Ake^{-k/k_c}$. The differences with a cold dark matter (CDM) model - which has the same linear Harrison-Zeldovich form at small k but a different (power-law) functional form for large k - are not fundamental here, and this PS allows us to calculate the correlation function ξ_c analytically (see GJSL02). This greatly simplifies our numerical calculation of the biased PS $P_\nu(k)$, which we do by direct integration of $\xi_\nu(r)$ calculated using the approximation

$$\xi_\nu(r) = \left[\sqrt{\frac{1+\xi_c(r)}{1-\xi_c(r)}}\exp\left(\nu^2\frac{\xi_c}{1+\xi_c}\right) - 1\right](1+o(\nu^{-1})) \tag{10}$$

which is very accurate over most of the range of the integration [2]. In Figure 1 we show $P_\nu(k)$ for various values of the threshold $\nu = 1, 2, 3$. We see that the shape of the PS at small k is completely changed with respect to the underlying PS. Indeed the main feature of the latter in this range - the display of a clear maximum and "turn-over" - is completely modified.

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[2]This approximation is obtained by expanding the full expression for $\xi_\nu(r)$ given in Eq. (4) in $1/\nu$, and further assuming only that $\nu\sqrt{(1-\xi_c)/(1+\xi_c)} \gg 1$. It is a much better approximation than that of Politzer and Wise both at small and larger values of ξ_c. In particular it gives an asymptotic behaviour $\xi_\nu \approx (\nu^2+1)\xi_c$ for $\nu^2\xi_c \ll 1$ which is a much better approximation to the exact behaviour at typically relevant values of ν ($b_1(1) \approx 2.4$)

Qualitatively it is not difficult to understand why this is so. The only characteristic scale in the PS (and also in the correlation function) is given by the turn-over (specified in our case by $k = k_c$). On the other hand, the value of $P_\nu(0)$ is just the integral over all space of ξ_ν which is proportional to the overall normalisation A and (since it is strictly positive) must be given on dimensional grounds by Ak_c times some function which depends on ν. For $\nu \simeq 1$ this function is of order one, so that $P_\nu(0) \sim max[P(k)]$.

This last point is better illustrated by considering the integral $J_3(r,\nu) = 4\pi \int_0^r x^2 \xi_\nu(x) dx$ which converges to $P_\nu(0) = \lim_{r\to\infty} J_3(r,\nu)$. In Figure 2 the value obtained for it by numerical integration of the *exact* expression given by Eq. (4) for $\xi_\nu(r)$ is shown for $\nu = 1, 2, 3$. We also show the same integral for ξ_c which converges to $P(0) = 0$. While the latter decreases at large r, converging very slowly to zero (as $1/r$ since $\xi_c(r) \propto -1/r^4$ at large scales), the former all converge towards a constant non-zero value We see that the integral picks up its dominant contribution from scales around (and above for $\nu = 1$) $r \sim 10$ Mpc (see caption for explanation of the normalisations, which are irrelevant for the present considerations). From the inset in the figure, which shows both $\xi_c(r)$ and $\xi_\nu(r)$, we see that this is the scale below which the correlation function is non-linearly amplified. Moreover it is shown that the smaller scales at which $\xi_\nu(r)$ is most distorted relative to $\xi_c(r)$ do not contribute significantly to J_3 (because of the r^2 factor). This fact also explains the accuracy of the PS obtained using the approximation Eq. (10) for $\xi_\nu(r)$, which can be seen by comparing the asymptotic values of the integrals in Figure 2 with $P_\nu(0)$ in Figure 1. Note that for $\nu = 1$ the distortion away from linear is relatively weak in the part of the correlation function which dominates the integral in $J_3(r,\nu)$, and that there is even a non-negligible contribution from the larger scales at which the correlation function amplification is extremely close to linear.

Let us now draw our conclusions. We have calculated the effect of the distortion at small k of a biased PS relative to an underlying PS, which we have shown to be an inevitable effect of biasing on cosmological models. In particular we have used a simplified model PS and the simplest (but reference) threshold biasing scheme of Kaiser. The latter is not a realistic model of biasing for various reasons, most notably because of the extremely strong non-linear distortion of the two-point correlation function at small scales whilst the observed ratio of correlation between galaxies and clusters is approximately linear. Our results however should be qualitatively correct for any biasing scheme and standard CDM type model. The only thing which we expect to depend on the biasing scheme is the functional form of the PS to the right of the turn-over (for $k > k_c$), and correspondingly the shape of the correlation function at small scales, which will only make a minor numerical change to our calculation. The essential feature of biasing which brings about the effect we have discussed - a non-linear amplification of the correlation function - will be common to any biasing model. In fact the converse of what we have argued is that any biasing scheme which is stochastic (i.e. any procedure for selecting sites for objects which is probabilistic) must give such a non-linear distortion of the correlation function for cosmological models: if the correlation function were amplified linearly at all scales, the asymptotic behaviour of the variance will be sub-Poissonian instead of Poissonian.

We finally draw two conclusions with respect to the comparison of cosmological models with observational data. In order to make the link to observations of galaxies or clusters, current theories make use normally of biasing in one form or another. Invariably however the measured PS from observations is fit to a model spectrum which is just a rescaled dark matter PS (for a recent example see Lahav et al. 2002). Our first conclusion is that such a behaviour cannot be obtained by biasing as the PS is not linearly amplified neither at small nor at large wavenumbers. Our second conclusion is that it will be very useful to measure not just the PS at small k, but also the real space correlation function at large distances. Invariably only the first is considered in analysis of observations of galaxy distributions at large scales, because, it is argued, it is the natural probe given that cosmological models of structure formation are formulated in k space. We have seen however that biasing leads to a distortion of the underlying dark matter PS, while (insert panel of Figure 2) the correlation function at sufficiently large distance remains undistorted. In particular for standard models with a H-Z PS at small k, the cleanest way to detect this behaviour in biased objects is by looking at the correlation function at large scales, which should maintain the behaviour $\xi_\nu(r) \sim -r^{-4}$ associated with the small k behaviour of the underlying dark matter PS. To address the viability of measuring this behaviour in current and forthcoming surveys of large scale structure requires in particular the detailed treatment of both the passage to the discrete distribution (we have treated here always continuous density fields), and the question of the variance of estimators of $\xi(r)$.

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Fig. 1.— The PS $P_\nu(k)$ derived from the biased correlation function $\xi_\nu(r)$ for values of the threshold $\nu = 1, 2, 3$ is shown. The underlying correlation function which gives $\xi_c(r)$ is that derived from the $P(k) = Ake^{-k/k_c}$ and the approximation given in Eq. (10) for ξ_ν is used. The clear distortion of the PS at small k is seen, the "turn-over" in the underlying PS essentially disappears already for $\nu = 1$. The constants k_c and A are fixed by $\xi_c(0) = 1$ and the requirement that $\xi_c(r) = 0$ at $r = 38$ Mpc. The latter is taken as in a typical CDM model (see e.g. Padmanabhan 1993). We could alternatively fix the wavenumber at the maximum of the PS.

Fig. 2.— The integral $J_3(r)$ for the same underlying correlation function as in Figure 1 and for the same range of values of ν, calculated with the exact expression Eq. (4) for $\xi_\nu(r)$. Also shown is the analagous integral of ξ_c. While the latter converges slowly to its asymptotic value of $P(0) = 0$, the other integrals converge to constant non-zero $P_\nu(0)$. They are dominated by the range $r \sim 10$ Mpc where, as can be seen from the inset which shows both ξ_c and each of the $\xi_\nu(r)$, the correlation functions $\xi_\nu(r)$ are amplified non-linearly. The contribution from the extremely amplified region at small r is small (because of the r^2 factor in the integral), which also makes the approximation in calculating the PS with Eq. (10) very accurate, as can be checked by comparing the asymptotic values with those of $P_\nu(0)$ in Figure 1.