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Abstract

A unified approach to (symmetric informationally complete) positive operator valued measures and mutually unbiased bases is developed in this article. The approach is based on the use of Racah unit tensors for the Wigner-Racah algebra of $\mathrm{SU}(2) \supset \mathrm{U}(1)$. Emphasis is put on similarities and differences between SIC-POVMs and MUBs.

Keywords: finite–dimensional Hilbert spaces; mutually unbiased bases; positive operator valued measures; $\mathrm{SU}(2) \supset \mathrm{U}(1)$ Wigner–Racah algebra

1 INTRODUCTION

The importance of finite–dimensional spaces for quantum mechanics is well recognized (see for instance [1]-[3]). In particular, such spaces play a major role in quantum information theory, especially for quantum cryptography and quantum state tomography [4]-[27]. Along this vein, a symmetric informationally complete (SIC) positive operator valued measure (POVM) is a set of operators acting on a finite Hilbert space [4]-[14] (see also [3] for an infinite Hilbert space) and mutually unbiased bases (MUBs) are specific bases for such a space [15]-[27].

The introduction of POVMs goes back to the seventies [4]-[7]. The most general quantum measurement is represented by a POVM. In the present work, we will be interested in SIC-POVMs, for which the statistics of the measurement allows the reconstruction of the quantum state. Moreover, those POVMs are endowed with an extra symmetry condition (see definition in Sec. 2). The notion of MUBs (see definition in Sec. 3), implicit or explicit in the seminal works of [15]-[18], has been the object of numerous mathematical and physical investigations during the last two decades in connection with the so-called complementary observables. Unfortunately, the question to know, for a given Hilbert space of finite dimension d, whether there exist SIC-POVMs and how many MUBs there exist has remained an open one.

The aim of this note is to develop a unified approach to SIC-POVMs and MUBs based on a complex vector space of higher dimension, viz. d^2 instead of d. We then give a specific example of this approach grounded on the Wigner-Racah algebra of the chain $\mathrm{SU}(2) \supset \mathrm{U}(1)$ recently used for a study of entanglement of rotationally invariant spin systems [28] and for an angular momentum study of MUBs [26, 27].

Most of the notations in this work are standard. Let us simply mention that $\mathbb{I}$ is the identity operator, the bar indicates complex conjugation, $A^\dagger$ denotes the adjoint of the operator A, $\delta_{a,b}$ stands for the Kronecker symbol for a and b, and $\Delta(a,b,c)$ is 1 or 0 according as a, b and c satisfy or not the triangular inequality.

2 SIC-POVMs

Let $\mathbb{C}^d$ be the standard Hilbert space of dimension d endowed with its usual inner product denoted by $\langle \mid \rangle$. As is usual, we will identify a POVM with a nonorthogonal decomposition of the identity. Thus, a discrete SIC-POVM is a set $\{P_x : x = 1, 2, \cdots, d^2\}$ of d^2 nonnegative operators P_x acting on $\mathbb{C}^d$, such that:

- they satisfy the *trace* or *symmetry condition*

$$\mathrm{Tr}\,(P_x P_y) = \frac{1}{d+1}, \quad x \neq y; \tag{1}$$

  moreover, we will assume the operators P_x are normalized, thus completing this condition with

$$\mathrm{Tr}\left(P_x^2\right) = 1; \tag{2}$$

- they form a *decomposition of the identity*

$$\frac{1}{d}\sum_{x=1}^{d^2} P_x = \mathbb{I}; \tag{3}$$

- they satisfy a *completeness condition*: the knowledge of the probabilities p_x defined by $p_x = \mathrm{Tr}(P_x \rho)$ is sufficient to reconstruct the density matrix ρ.

Now, let us develop each of the operators P_x on an orthonormal (with respect to the Hilbert–Schmidt product) basis $\{u_i : i = 1, 2, \cdots, d^2\}$ of the space of linear operators on $\mathbb{C}^d$

$$P_x = \sum_{i=1}^{d^2} v_i(x) u_i, \tag{4}$$

where the operators u_i satisfy $\mathrm{Tr}(u_i^\dagger u_j) = \delta_{i,j}$. The operators P_x are thus considered as vectors

$$v(x) = (v_1(x), v_2(x), \cdots, v_{d^2}(x)) \tag{5}$$

in the Hilbert space $\mathbb{C}^{d^2}$ of dimension d^2 and the determination of the operators P_x is equivalent to the determination of the components $v_i(x)$ of $v(x)$. In this language, the trace property (1) together with the normalization condition (2) give

$$v(x) \cdot v(y) = \frac{1}{d+1}\left(d\delta_{x,y} + 1\right), \tag{6}$$

where $v(x) \cdot v(y) = \sum_{i=1}^{d^2} \overline{v_i(x)} v_i(y)$ is the usual Hermitian product in $\mathbb{C}^{d^2}$.

In order to compare Eq. (6) with what usually happens in the search for SIC-POVMs, we suppose from now on that the operators P_x are rank-one operators. Therefore, by putting

$$P_x = |\Phi_x\rangle\langle\Phi_x| \tag{7}$$

with $|\phi_x\rangle \in \mathbb{C}^d$, the trace property (1, 2) reads

$$|\langle \Phi_x | \Phi_y \rangle|^2 = \frac{1}{d+1} \left(d\delta_{x,y} + 1 \right). \tag{8}$$

From this point of view, to find d^2 operators P_x is equivalent to finding d^2 vectors $|\phi_x\rangle$ in $\mathbb{C}^d$ satisfying Eq. (8). At the price of an increase in the number of components from d^3 (for d^2 vectors in $\mathbb{C}^d$) to d^4 (for d^2 vectors in $\mathbb{C}^{d^2}$), we have got rid of the square modulus to result in a single scalar product (compare Eqs. (6) and (8)), what may prove to be suitable for another way to search for SIC-POVMs. Moreover, our relation (6) is independent of any hypothesis on the rank of the operators P_x. In fact, there exists a lot of relations among these d^4 coefficients that decrease the effective number of coefficients to be found and give structural constraints on them. Those relations are highly sensitive to the choice of the basis $\{u_i : i = 1, 2, \cdots, d^2\}$ and we are going to exhibit an example of such a set of relations by choosing the basis to consist of Racah unit tensors.

The cornerstone of this approach is to identify $\mathbb{C}^d$ with a subspace $\varepsilon(j)$ of constant angular momentum $j = (d-1)/2$. Such a subspace is spanned by the set $\{|j,m\rangle : m = -j, -j+1, \cdots, j\}$, where $|j,m\rangle$ is an eigenvector of the square and the z-component of a generalized angular momentum operator. Let $\mathbf{u}^{(k)}$ be the Racah unit tensor [29] of order k (with $k = 0, 1, \cdots, 2j$) defined by its $2k+1$ components $u_q^{(k)}$ (where $q = -k, -k+1, \cdots, k$) through

$$u_q^{(k)} = \sum_{m=-j}^{j} \sum_{m'=-j}^{j} (-1)^{j-m} \begin{pmatrix} j & k & j \\ -m & q & m' \end{pmatrix} |j,m\rangle\langle j,m'|, \tag{9}$$

where $(\cdots)$ denotes a 3–jm Wigner symbol. For fixed j, the $(2j+1)^2$ operators $u_q^{(k)}$ (with $k = 0, 1, \cdots, 2j$ and $q = -k, -k+1, \cdots, k$) act on $\varepsilon(j) \sim \mathbb{C}^d$ and form a basis of the Hilbert space $\mathbb{C}^N$ of dimension $N = (2j+1)^2$, the inner product in $\mathbb{C}^N$ being the Hilbert–Schmidt product. The formulas (involving unit tensors, 3–jm and 6–j symbols) relevant for this work are given in Appendix (see also [29] to [31]). We must remember that those Racah operators are not normalized to unity (see relation (46)). So this will generate an extra factor when defining $v_i(x)$.

Each operator P_x can be developed as a linear combination of the operators $u_q^{(k)}$. Hence, we have

$$P_x = \sum_{k=0}^{2j} \sum_{q=-k}^{k} c_{kq}(x) u_q^{(k)}, \tag{10}$$

where the unknown expansion coefficients $c_{kq}(x)$ are *a priori* complex numbers. The determination of the operators P_x is thus equivalent to the determination of the coefficients $c_{kq}(x)$, which are formally given by

$$c_{kq}(x) = (2k+1)\overline{\langle \Phi_x | u_q^{(k)} | \Phi_x \rangle}, \tag{11}$$

as can be seen by multiplying each member of Eq. (10) by the adjoint of $u_p^{(\ell)}$ and then using Eq. (46) of Appendix.

By defining the vector

$$v(x) = (v_1(x), v_2(x), \cdots, v_N(x)), \quad N = (2j+1)^2 \tag{12}$$

via

$$v_i(x) = \frac{1}{\sqrt{2k+1}} c_{kq}(x), \quad i = k^2 + k + q + 1, \tag{13}$$

the following properties and relations are obtained.

- *The first component* $v_1(x)$ *of* $v(x)$ *does not depend on* x since

  $$c_{00}(x) = \frac{1}{\sqrt{2j+1}} \tag{14}$$

  for all $x \in \{1, 2, \cdots, (2j+1)^2\}$.

  Proof: Take the trace of Eq. (10) and use Eq. (48) of Appendix.

- The components $v_i(x)$ of $v(x)$ satisfy the *complex conjugation property* described by

  $$\overline{c_{kq}(x)} = (-1)^q c_{k-q}(x) \tag{15}$$

  for all $x \in \{1, 2, \cdots, (2j+1)^2\}$, $k \in \{0, 1, \cdots, 2j\}$ and $q \in \{-k, -k+1, \cdots, k\}$.

  Proof: Use the Hermitian property of P_x and Eq. (43) of Appendix.

- In terms of c_{kq}, Eq. (6) reads

$$\sum_{k=0}^{2j} \frac{1}{2k+1} \sum_{q=-k}^{k} \overline{c_{kq}(x)} c_{kq}(y) = \frac{1}{2(j+1)} \left[(2j+1)\delta_{x,y} + 1\right] \tag{16}$$

for all $x, y \in \{1, 2, \cdots, (2j+1)^2\}$, where the sum over q is $\mathrm{SO}(3)$ rotationally invariant.

Proof: The proof is trivial.

- The coefficients $c_{kq}(x)$ are solutions of the *nonlinear system* given by

$$\begin{aligned} \frac{1}{2K+1} c_{KQ}(x) &= (-1)^{2j-Q} \sum_{k=0}^{2j} \sum_{\ell=0}^{2j} \sum_{q=-k}^{k} \sum_{p=-\ell}^{\ell} \begin{pmatrix} k & \ell & K \\ -q & -p & Q \end{pmatrix} \\ &\times \begin{Bmatrix} k & \ell & K \\ j & j & j \end{Bmatrix} c_{kq}(x) c_{\ell p}(x) \end{aligned} \tag{17}$$

for all $x \in \{1, 2, \cdots, (2j+1)^2\}$, $K \in \{0, 1, \cdots, 2j\}$ and $Q \in \{-K, -K+1, \cdots, K\}$.

Proof: Consider $P_x^2 = P_x$ and use the coupling relation (51) of Appendix involving a 3–jm and a 6–j Wigner symbols.

As a corollary of the latter property, by taking $K = 0$ and using Eqs. (47) and (50) of Appendix, we get again the normalization relation $\|v(x)\|^2 = v(x) \cdot v(x) = 1$.

- All coefficients $c_{kq}(x)$ are connected through the *sum rule*

$$\sum_{x=1}^{(2j+1)^2} \sum_{k=0}^{2j} \sum_{q=-k}^{k} c_{kq}(x) \begin{pmatrix} j & k & j \\ -m & q & m' \end{pmatrix} = (-1)^{j-m}(2j+1)\delta_{m,m'}, \tag{18}$$

which turns out to be useful for global checking purposes.

Proof: Take the jm–jm' matrix element of the resolution of the identity in terms of the operators $P_x/(2j+1)$.

3 MUBs

A complete set of MUBs in the Hilbert space $\mathbb{C}^d$ is a set of $d(d+1)$ vectors $|a\alpha\rangle \in \mathbb{C}^d$ such that

$$|\langle a\alpha|b\beta\rangle|^2 = \delta_{\alpha,\beta}\delta_{a,b} + \frac{1}{d}(1 - \delta_{a,b}), \tag{19}$$

where $a = 0, 1, \cdots, d$ and $\alpha = 0, 1, \cdots, d-1$. The indices of type a refer to the bases and, for fixed a, the index α refers to one of the d vectors of the basis corresponding to a. We know that such a complete set exists if d is a prime or the power of a prime (*e.g.*, see [16]-[24]).

The approach developed in Sec. 2 for SIC-POVMs can be applied to MUBs too. Let us suppose that it is possible to find $d+1$ sets S_a (with $a = 0, 1, \cdots, d$) of vectors in $\mathbb{C}^d$, each set $S_a = \{|a\alpha\rangle : \alpha = 0, 1, \cdots, d-1\}$ containing d vectors $|a\alpha\rangle$ such that Eq. (19) be satisfied. This amounts to finding $d(d+1)$ *projection operators*

$$\Pi_{a\alpha} = |a\alpha\rangle\langle a\alpha| \tag{20}$$

satisfying the *trace condition*

$$\mathrm{Tr}\,(\Pi_{a\alpha}\Pi_{b\beta}) = \delta_{\alpha,\beta}\delta_{a,b} + \frac{1}{d}(1 - \delta_{a,b}), \tag{21}$$

where the trace is taken on $\mathbb{C}^d$. Therefore, they also form a *nonorthogonal decomposition of the identity*

$$\frac{1}{d+1}\sum_{a=0}^{d}\sum_{\alpha=0}^{d-1}\Pi_{a\alpha} = \mathbb{I}. \tag{22}$$

As in Sec. 2, we develop each operator $\Pi_{a\alpha}$ on an orthonormal basis with expansion coefficients $w_i(a\alpha)$. Thus we get vectors $w(a\alpha)$ in $\mathbb{C}^{d^2}$

$$w(a\alpha) = (w_1(a\alpha), w_2(a\alpha), \cdots, w_{d^2}(a\alpha)) \tag{23}$$

such that

$$w(a\alpha) \cdot w(b\beta) = \delta_{\alpha,\beta}\delta_{a,b} + \frac{1}{d}(1 - \delta_{a,b}) \tag{24}$$

for all $a, b \in \{0, 1, \cdots, d\}$ and $\alpha, \beta \in \{0, 1, \cdots, d-1\}$.

Now we draw the same relations as for POVMs by choosing the Racah operators to be our basis in $\mathbb{C}^{d^2}$. We assume once again that the Hilbert space $\mathbb{C}^d$ is realized by $\varepsilon(j)$ with $j = (d-1)/2$. Then, each operator $\Pi_{a\alpha}$ can be developed on the basis of the $(2j+1)^2$

operators $u_q^{(k)}$ as

$$\Pi_{a\alpha} = \sum_{k=0}^{2j} \sum_{q=-k}^{k} d_{kq}(a\alpha) u_q^{(k)}, \tag{25}$$

to be compared with Eq. (10). The expansion coefficients are

$$d_{kq}(a\alpha) = (2k+1)\overline{\langle a\alpha | u_q^{(k)} | a\alpha \rangle} \tag{26}$$

for all $a \in \{0, 1, \cdots, 2j+1\}$, $\alpha \in \{0, 1, \cdots, 2j\}$, $k \in \{0, 1, \cdots, 2j\}$ and $q \in \{-k, -k+1, \cdots, k\}$. For a and α fixed, the complex coefficients $d_{kq}(a\alpha)$ define a vector

$$w(a\alpha) = (w_1(a\alpha), w_2(a\alpha), \cdots, w_N(a\alpha)), \quad N = (2j+1)^2 \tag{27}$$

in the Hilbert space $\mathbb{C}^N$, the components of which are given by

$$w_i(a\alpha) = \frac{1}{\sqrt{2k+1}} d_{kq}(a\alpha), \quad i = k^2 + k + q + 1. \tag{28}$$

We are thus led to the following properties and relations. The proofs are similar to those in Sec. 2.

- *First component* $w_1(a\alpha)$ *of* $w(a\alpha)$:

$$d_{00}(a\alpha) = \frac{1}{\sqrt{2j+1}} \tag{29}$$

  for all $a \in \{0, 1, \cdots, 2j+1\}$ and $\alpha \in \{0, 1, \cdots, 2j\}$.

- *Complex conjugation property*:

$$\overline{d_{kq}(a\alpha)} = (-1)^q d_{k-q}(a\alpha) \tag{30}$$

  for all $a \in \{0, 1, \cdots, 2j+1\}$, $\alpha \in \{0, 1, \cdots, 2j\}$, $k \in \{0, 1, \cdots, 2j\}$ and $q \in \{-k, -k+1, \cdots, k\}$.

- *Rotational invariance*:

$$\sum_{k=0}^{2j} \frac{1}{2k+1} \sum_{q=-k}^{k} \overline{d_{kq}(a\alpha)} d_{kq}(b\beta) = \delta_{\alpha,\beta}\delta_{a,b} + \frac{1}{2j+1}(1 - \delta_{a,b}) \tag{31}$$

  for all $a, b \in \{0, 1, \cdots, 2j+1\}$ and $\alpha, \beta \in \{0, 1, \cdots, 2j\}$.

- *Tensor product formula*:

$$\frac{1}{2K+1}d_{KQ}(a\alpha) = (-1)^{2j-Q}\sum_{k=0}^{2j}\sum_{\ell=0}^{2j}\sum_{q=-k}^{k}\sum_{p=-\ell}^{\ell}\begin{pmatrix} k & \ell & K \\ -q & -p & Q \end{pmatrix} \times \begin{Bmatrix} k & \ell & K \\ j & j & j \end{Bmatrix} d_{kq}(a\alpha)d_{\ell p}(a\alpha) \tag{32}$$

for all $a \in \{0, 1, \cdots, 2j+1\}$, $\alpha \in \{0, 1, \cdots, 2j\}$, $K \in \{0, 1, \cdots, 2j\}$ and $Q \in \{-K, -K+1, \cdots, K\}$.

- *Sum rule*:

$$\sum_{a=0}^{2j+1}\sum_{\alpha=0}^{2j}\sum_{k=0}^{2j}\sum_{q=-k}^{k} d_{kq}(a\alpha)\begin{pmatrix} j & k & j \\ -m & q & m' \end{pmatrix} = (-1)^{j-m}2(2j+1)\delta_{m,m'} \tag{33}$$

which involves all coefficients $d_{kq}(a\alpha)$.

4 CONCLUSIONS

Although the structure of the relations in Sec. 1 on the one hand and Sec. 2 on the other hand is very similar, there are deep differences between the two sets of results. The similarities are reminiscent of the fact that both MUBs and SIC-POVMs can be linked to finite affine planes [12, 13, 22, 23, 25] and to complex projective 2–designs [8, 10, 19, 24]. On the other side, there are two arguments in favor of the differences between relations (6) and (24). First, the problem of constructing SIC-POVMs in dimension d is not equivalent to the existence of an affine plane of order d [12, 13]. Second, there is a consensus around the conjecture according to which there exists a complete set of MUBs in dimension d if and only if there exists an affine plane of order d [22].

In dimension d, to find d^2 operators P_x of a SIC-POVM acting on the Hilbert space $\mathbb{C}^d$ amounts to find d^2 vectors $v(x)$ in the Hilbert space $\mathbb{C}^N$ with $N = d^2$ satisfying

$$\|v_x\| = 1, \quad v(x)\cdot v(y) = \frac{1}{d+1} \text{ for } x \neq y \tag{34}$$

(the norm $\|v(x)\|$ of each vector $v(x)$ is 1 and the angle ω_{xy} of any pair of vectors $v(x)$ and $v(y)$ is $\omega_{xy} = \cos^{-1}[1/(d+1)]$ for $x \neq y$).

In a similar way, to find $d+1$ MUBs of $\mathbb{C}^d$ is equivalent to find $d+1$ sets S_a (with $a = 0, 1, \cdots, d$) of d vectors, *i.e.*, $d(d+1)$ vectors in all, $w(a\alpha)$ in $\mathbb{C}^N$ with $N = d^2$ satisfying

$$w(a\alpha) \cdot w(a\beta) = \delta_{\alpha,\beta}, \quad w(a\alpha) \cdot w(b\beta) = \frac{1}{d} \text{ for } a \neq b \tag{35}$$

(each set S_a consists of d orthonormalized vectors and the angle $\omega_{a\alpha b\beta}$ of any vector $w(a\alpha)$ of a set S_a with any vector $w(b\beta)$ of a set S_b is $\omega_{a\alpha b\beta} = \cos^{-1}(1/d)$ for $a \neq b$).

According to a well accepted conjecture [8, 10], SIC-POVMs should exist in any dimension. The present study shows that in order to prove this conjecture it is sufficient to prove that Eq. (34) admits solutions for any value of d.

The situation is different for MUBs. In dimension d, it is known that there exist $d+1$ sets of d vectors of type $|a\alpha\rangle$ in $\mathbb{C}^d$ satisfying Eq. (19) when d is a prime or the power of a prime. This shows that Eq. (35) can be solved for d prime or power of a prime. For d prime, it is possible to find an explicit solution of Eq. (19). In fact, we have [26, 27]

$$|a\alpha\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} \omega^{(j+m)(j-m+1)a/2+(j+m)\alpha} |j,m\rangle, \tag{36}$$

$$\omega = \exp\left(\mathrm{i}\frac{2\pi}{2j+1}\right), \quad j = \frac{1}{2}(d-1) \tag{37}$$

for $a, \alpha \in \{0, 1, \cdots, 2j\}$ while

$$|a\alpha\rangle = |j,m\rangle \tag{38}$$

for $a = 2j+1$ and $\alpha = j+m = 0, 1, \cdots, 2j$. Then, Eq. (26) yields

$$d_{kq}(a\alpha) = \frac{2k+1}{2j+1} \sum_{m=-j}^{j} \sum_{m'=-j}^{j} \omega^{\theta(m,m')} (-1)^{j-m} \begin{pmatrix} j & k & j \\ -m & q & m' \end{pmatrix}, \tag{39}$$

$$\theta(m,m') = (m-m')\left[\frac{1}{2}(1-m-m')a + \alpha\right] \tag{40}$$

for $a, \alpha \in \{0, 1, \cdots, 2j\}$ while

$$d_{kq}(a\alpha) = \delta_{q,0}(2k+1)(-1)^{j-m} \begin{pmatrix} j & k & j \\ -m & 0 & m \end{pmatrix} \tag{41}$$

for $a = 2j+1$ and $\alpha = j+m = 0, 1, \cdots, 2j$. It can be shown that Eqs. (40) and (41) are in agreement with the results of Sec. 3. We thus have a solution of the equations for the results of Sec. 3 when d is prime. As an open problem, it would be worthwhile to find an explicit solution for the coefficients $d_{kq}(a\alpha)$ when $d = 2j + 1$ is any positive power of a prime. Finally, note that to prove (or disprove) the conjecture according to which a complete set of MUBs in dimension d exists only if d is a prime or the power of a prime is equivalent to prove (or disprove) that Eq. (35) has a solution only if d is a prime or the power of a prime.

APPENDIX: WIGNER-RACAH ALGEBRA OF $\mathrm{SU}(2) \supset \mathrm{U}(1)$

We limit ourselves to those basic formulas for the Wigner-Racah algebra of the chain $\mathrm{SU}(2) \supset \mathrm{U}(1)$ which are necessary to derive the results of this paper. The summations in this appendix have to be extended to the allowed values for the involved magnetic and angular momentum quantum numbers.

The definition (9) of the components $u_q^{(k)}$ of the Racah unit tensor $\mathbf{u}^{(k)}$ yields

$$\langle j, m | u_q^{(k)} | j, m' \rangle = (-1)^{j-m} \begin{pmatrix} j & k & j \\ -m & q & m' \end{pmatrix}, \tag{42}$$

from which we easily obtain the Hermitian conjugation property

$$u_q^{(k)\dagger} = (-1)^q u_{-q}^{(k)}. \tag{43}$$

The 3–jm Wigner symbol in Eq. (42) satisfies the orthogonality relations

$$\sum_{mm'} \begin{pmatrix} j & j' & k \\ m & m' & q \end{pmatrix} \begin{pmatrix} j & j' & \ell \\ m & m' & p \end{pmatrix} = \frac{1}{2k+1} \delta_{k,\ell} \delta_{q,p} \Delta(j, j', k) \tag{44}$$

and

$$\sum_{kq} (2k+1) \begin{pmatrix} j & j' & k \\ m & m' & q \end{pmatrix} \begin{pmatrix} j & j' & k \\ M & M' & q \end{pmatrix} = \delta_{m,M} \delta_{m',M'}. \tag{45}$$

The trace relation on the space $\varepsilon(j)$

$$\mathrm{Tr} \left(u_q^{(k)\dagger} u_p^{(\ell)} \right) = \frac{1}{2k+1} \delta_{k,\ell} \delta_{q,p} \Delta(j, j, k) \tag{46}$$

easily follows by combining Eqs. (42) and (44). Furthermore, by introducing

$$\begin{pmatrix} j & j' & 0 \\ m & -m' & 0 \end{pmatrix} = \delta_{j,j'}\delta_{m,m'}(-1)^{j-m}\frac{1}{\sqrt{2j+1}} \tag{47}$$

in Eq. (44), we obtain the sum rule

$$\sum_m (-1)^{j-m} \begin{pmatrix} j & k & j \\ -m & q & m \end{pmatrix} = \sqrt{2j+1}\delta_{k,0}\delta_{q,0}\Delta(j,k,j), \tag{48}$$

known in spectroscopy as the barycenter theorem.

There are several relations involving 3–jm and 6–j symbols. In particular, we have

$$\begin{aligned} \sum_{mm'M} \quad & (-1)^{j-M} \begin{pmatrix} j & k & j \\ -m & q & M \end{pmatrix} \begin{pmatrix} j & \ell & j \\ -M & p & m' \end{pmatrix} \begin{pmatrix} j & K & j \\ -m & Q & m' \end{pmatrix} \\ = \quad & (-1)^{2j-Q} \begin{pmatrix} k & \ell & K \\ -q & -p & Q \end{pmatrix} \begin{Bmatrix} k & \ell & K \\ j & j & j \end{Bmatrix}, \end{aligned} \tag{49}$$

where $\{\cdots\}$ denotes a 6–j Wigner symbol (or $\overline{W}$ Racah coefficient). Note that the introduction of

$$\begin{Bmatrix} k & \ell & 0 \\ j & j & J \end{Bmatrix} = \delta_{k,\ell}(-1)^{j+k+J}\frac{1}{\sqrt{(2k+1)(2j+1)}} \tag{50}$$

in Eq. (49) gives back Eq. (44). Equation (49) is central in the derivation of the coupling relation

$$u_q^{(k)}u_p^{(\ell)} = \sum_{KQ}(-1)^{2j-Q}(2K+1)\begin{pmatrix} k & \ell & K \\ -q & -p & Q \end{pmatrix}\begin{Bmatrix} k & \ell & K \\ j & j & j \end{Bmatrix}u_Q^{(K)}. \tag{51}$$

Equation (51) makes it possible to calculate the commutator $[u_q^{(k)}, u_p^{(\ell)}]$ which shows that the set $\{u_q^{(k)} : k = 0, 1, \cdots, 2j; \quad q = -k, -k+1, \cdots, k\}$ can be used to span the Lie algebra of the unitary group U($2j+1$). The latter result is at the root of the expansions (17) and (32).

Note added in version 3

After the submission of the present paper for publication in *Journal of Russian Laser Research*, a pre-print dealing with the existence of SIC-POVMs was posted on arXiv [32]. The main result in [32] is that SIC-POVMs exist in all dimensions. As a corollary of this result, Eq. (34) admits solutions in any dimension.

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