Continued fraction approximation for the nuclear matter response function

Jérôme Margueron, J. Navarro, Nguyen van Giai, P. Schuck

Continued fraction approximation for the nuclear matter response function

J. Margueron,[1] J. Navarro,[2] Nguyen Van Giai,[1] and P. Schuck[1]

[1]*Institut de Physique Nucléaire (UMR 8608), CNRS/IN2P3-Université Paris-Sud, F-91406 ORSAY CEDEX, France*
[2]*IFIC (CSIC - Universidad de Valencia), Apdo. 22085, E-46.071-Valencia, Spain*

(Dated: February 21, 2008)

We use a continued fraction approximation to calculate the RPA response function of nuclear matter. The convergence of the approximation is assessed by comparing with the numerically exact response function obtained with a typical effective finite-range interaction used in nuclear physics. It is shown that just the first order term of the expansion can give reliable results at densities up to the saturation density value.

PACS numbers: 21.30.-x, 21.60.Jz, 21.65.-f
Keywords: effective nuclear interactions, nuclear matter, response functions, random phase approximation

I. INTRODUCTION

There are many physical issues that require the knowledge of the response function of a medium to an external probe. Well-known examples are the electron scattering by nuclei or the propagation of neutrinos in nuclear matter. In order to develop a microscopic theory of response functions in finite nuclear systems one usually starts by considering the limiting case of an infinite medium. Infinite nuclear matter as a homogeneous medium made of interacting nucleons is a very useful and broadly used concept because of its relative simplicity and its connection with the bulk part of atomic nuclei. A popular approach consists in using an effective nucleon-nucleon interaction adjusted to describe the nuclear matter properties in a mean field approximation. Then, this microscopic description can be extended to finite nuclei.

In a mean field framework the nuclear response functions must take into account the effects of long-range correlations by the Random Phase Approximation (RPA) which is the small amplitude limit of a time-dependent mean field approach. This is well suited for those excitations which correspond to small amplitude vibrations, the most typical of which being the giant resonances and the low-lying collective states [1]. For the theory to be consistent, it is necessary that the same effective nucleon-nucleon interaction generates the self-consistent Hartree-Fock (HF) mean field and the RPA correlations which lead to the excitations of the system.

There are two types of interactions widely used in non-relativistic approaches, the zero-range Skyrme-type forces [2] and the finite-range Gogny-type forces [3]. Skyrme forces are very often used because of their relatively simple analytic form which allows for quite complete RPA calculations in nuclear matter [4] as well as in finite nuclei [5, 6]. On the other hand, finite-range forces require heavier computational efforts to calculate RPA responses in nuclei [7, 8]. Furthermore, the only existing methods in this case consist in diagonalizing large size matrices in configuration space. It would be useful to have alternative methods such as a direct calculation in coordinate space or momentum space of RPA response functions, to avoid the increasingly large configuration spaces. This is possible with Skyrme forces [9] but in the case of finite range forces the exchange interactions complicate the problem.

In this work we study an approximation based on a continued fraction expansion of the response function. Our aim is to explore a calculational scheme which can be checked in infinite matter and which offers prospects for RPA calculations with finite range forces in nuclei. The continued fraction method is known in the literature [10] and it has been used by many authors to study response functions in the quasi-elastic regime (see Ref. [11] and references therein). However, it is difficult to know where to truncate the continued fraction expansion to obtain a desired accuracy. It is possible to calculate response functions in infinite matter by performing multipole expansions of the interaction and to have numerically accurate results [12] to evaluate various approximation schemes. Therefore, the present study aims at assessing the speed of convergence of the continued fraction expansion applied to the response functions in nuclear matter, using as an example a Gogny force D1 [3]. We show that this expansion gives good results as compared with the numerically exact calculations, even at lowest order.

In Sec.II we recall the basic features of the continued fraction method applied to the determination of RPA response functions in an infinite medium, and we show analytically that it gives the correct result in the special case of a Landau-Migdal interaction. In Sec.III we discuss the results obtained with a finite range interaction of Gogny type. Conclusions are drawn in Sec.IV .

II. FORMALISM

A. General framework

A general two-body interaction in momentum representation depends at most on 4 momenta. Because of momentum conservation there are actually 3 independent momenta, in the case of a translationally invariant interaction. For the particle-hole (p-h) case we choose these independent variables to be the initial (final) momentum $\mathbf{k_1}$ ($\mathbf{k_2}$) of the hole and the external momentum transfer $\mathbf{q}$. We follow the notations of Ref. [12] and we denote by $\alpha = (S, M; T, Q)$ the spin and isospin p-h channels with S=0 (1) for the non spin-flip (spin-flip) channel, T=0 (1) the isoscalar (isovector) channel, M and Q being the third components of S and T. The matrix element of the general p-h interaction including exchange can be written as:

$$V_{\text{ph}}^{(\alpha,\alpha')}(\mathbf{q},\mathbf{k}_1,\mathbf{k}_2) \equiv \langle \mathbf{q}+\mathbf{k}_1, \mathbf{k}_1^{-1}, (\alpha)|V|\mathbf{q}+\mathbf{k}_2, \mathbf{k}_2^{-1}, (\alpha')\rangle \ . \tag{1}$$

To calculate the response of a homogeneous medium to an external field it is convenient to introduce the Green's function, or retarded p-h propagator $G^{(\alpha)}(\mathbf{q}, \omega, \mathbf{k}_1)$. From now on we choose the z axis along the direction of $\mathbf{q}$. In the HF approximation the p-h Green's function is the free retarded p-h propagator [13]:

$$G_{\text{HF}}(\text{q}, \omega, \mathbf{k}_1) = \frac{f(k_1) - f(|\mathbf{k}_1+\mathbf{q}|)}{\omega + \epsilon(k_1) - \epsilon(|\mathbf{k}_1+\mathbf{q}|) + \text{i}\eta} \ , \tag{2}$$

where $\epsilon(k)$ is the HF single-particle energy corresponding to momentum $\mathbf{k}$, and the Fermi-Dirac distribution f is defined for a given temperature T and chemical potential μ as $f(k) = [1 + e^{(\epsilon(k)-\mu)/T}]^{-1}$. The HF Green's function G_{HF} does not depend on the spin-isospin channel α.

To go beyond the HF mean field approximation one takes into account the long-range type of correlations by re-summing a class of p-h diagrams. One thus obtains the well-known RPA [13] whose correlated Green's function $G^{(\alpha)}_{\text{RPA}}(\text{q}, \omega, \mathbf{k}_1)$ satisfies the Bethe-Salpeter equation:

$$G^{(\alpha)}_{\text{RPA}}(\text{q}, \omega, \mathbf{k}_1) = G_{\text{HF}}(\text{q}, \omega, \mathbf{k}_1) + G_{\text{HF}}(\text{q}, \omega, \mathbf{k}_1) \sum_{(\alpha')} \int \frac{\text{d}^3 k_2}{(2\pi)^3} V_{\text{ph}}^{(\alpha,\alpha')}(\text{q}, \mathbf{k}_1, \mathbf{k}_2) G^{(\alpha')}_{\text{RPA}}(\text{q}, \omega, \mathbf{k}_2) \ . \tag{3}$$

Finally, the response function $\chi^{(\alpha)}(\text{q}, \omega)$ in the infinite medium is related to the p-h Green's function by:

$$\chi^{(\alpha)}_{\text{RPA}}(\text{q}, \omega) = g \int \frac{\text{d}^3 k_1}{(2\pi)^3} G^{(\alpha)}_{\text{RPA}}(\text{q}, \omega, \mathbf{k}_1) \ , \tag{4}$$

where the spin-isospin degeneracy factor g is 4 in symmetric nuclear matter and 2 in pure neutron matter. In the case of a system of particles without residual interactions the free response is obtained by calculating Eq. (4) with the HF p-h propagator G_{HF}, thus obtaining the well-known Lindhard function χ_{HF}.

B. Continued fraction approximation

A direct numerical solution of Eq. (3) with a general p-h interaction is possible, as it has been shown in Ref. [12] for the Gogny interaction. However, such a method is specifically designed for infinite systems and it would be interesting to have an alternative method which can be accurate and at the same time can be used in calculations of finite systems. We examine now an approximate way to calculate the RPA response function, expressing it as a continued fraction. To simplify the writing of the equations we shall employ the following conventions. First of all we omit the variables such as q, ω or $\mathbf{k}$ as well as indices (α), unless necessary. For instance, equation (3) is written as

$$G_{\text{RPA}} = G_{\text{HF}} + G_{\text{HF}} V_{\text{ph}} G_{\text{RPA}} \ . \tag{5}$$

Secondly, for any function $F(\mathbf{k}_1)$ depending on a momentum $\mathbf{k}_1$ we denote by $\langle F\rangle$ its integrated value over momentum space. For example,

$$\langle G_{\text{HF}}\rangle \equiv \int \frac{d^3 k_1}{(2\pi)^3} G_{\text{HF}}(\mathbf{k}_1) \ , \tag{6}$$

so that Eq. (4) is simply written as

$$\chi_{\rm RPA} = g\langle G_{\rm RPA}\rangle \ . \tag{7}$$

The Bethe-Salpeter equation is an integral equation which can, in principle, be solved iteratively

$$G_{\rm RPA} = G_{\rm HF} + G_{\rm HF}V_{\rm ph}G_{\rm HF} + G_{\rm HF}V_{\rm ph}G_{\rm HF}V_{\rm ph}G_{\rm HF} + \ldots \tag{8}$$

Correspondingly, the RPA response function is written as

$$\chi_{\rm RPA} = \chi_{\rm HF} + g\langle G_{\rm HF}(1)V_{\rm ph}(1,2)G_{\rm HF}(2)\rangle + g\langle G_{\rm HF}(1)V_{\rm ph}(1,2)G_{\rm HF}(2)V_{\rm ph}(2,3)G_{\rm HF}(3)\rangle + \cdots \tag{9}$$

The brackets imply integrations over chains of variables as shown here.

In Ref. [10] an approximation was suggested by defining an effective interaction $V_{\rm eff}({\rm q},\omega,T)$ such that the RPA response function is written as

$$\chi_{\rm RPA} = \frac{\chi_{\rm HF}}{1 - V_{\rm eff}\chi_{\rm HF}} \ . \tag{10}$$

In the RPA neglecting exchange (the ring approximation) the effective interaction does not depend on the hole momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ so that Eq. (10) is exact if one replaces $V_{\rm eff}$ by $V_{\rm ph}$. However, it is important to treat direct and exchange terms on equal footing, since they are in general of the same order of magnitude. Here, our point of view differs from other works where the direct and exchange interactions are treated on different approximation levels [11]. We express the effective interaction as a continued fraction

$$V_{\rm eff} = \cfrac{V_1}{1 - \cfrac{V_2\chi_{\rm HF}}{1 - \cfrac{V_3\chi_{\rm HF}}{1 - \ldots}}} \ . \tag{11}$$

Each term V_i entering this definition is deduced by expanding formally Eqs. (11) and (10) in powers of $V_i\chi_{\rm HF}$ and identifying with Eq. (9). The explicit expression for the first two terms are:

$$\begin{aligned} V_1 &= \frac{g\langle G_{\rm HF}V_{\rm ph}G_{\rm HF}\rangle}{(\chi_{\rm HF})^2} \ , \\ V_2 &= \frac{g\langle G_{\rm HF}V_{\rm ph}G_{\rm HF}V_{\rm ph}G_{\rm HF}\rangle}{V_1 \ (\chi_{\rm HF})^3} - V_1 \ . \end{aligned} \tag{12}$$

First, one can notice that the quantities $G_{\rm HF}$ and $\chi_{\rm HF}$ are complex functions of q, ω and T, and so are the V_i and the effective interaction $V_{\rm eff}$. Second, the calculations of the V_i in the infinite medium involve only products of functions, which is somewhat easier numerically than the full calculations of response functions where one needs to perform matrix inversions [12]. Third, one see that V_1 is just the average of the full p-h interaction over the squared free p-h Green's function. Therefore, the continued fraction approximation could be quite useful for calculating RPA response functions if one checks how accurate it can be for a general interaction like the Gogny force. This is what we shall examine in Sec. III.

C. An analytical case: the Landau-Migdal interaction

The convergence of the approximation can be explicitly seen in the schematic case of a p-h interaction of the Landau-Migdal form containing $\ell = 0$ and $\ell = 1$ terms:

$$V_{\rm ph} = g\left\{f_0 + f_1\cos\theta_{12}\right\} \tag{13}$$

where for brevity the same notation f_i is used for the Landau parameters in the four spin-isospin channels. For such an interaction the RPA response function can be analytically calculated (see e.g. Ref. 4):

$$\chi_{\rm RPA} = \frac{\chi_{\rm HF}}{1 - \left(f_0 + \dfrac{f_1\nu^2}{1+F_1/3}\right)\chi_{\rm HF}} \ , \tag{14}$$

where $\nu = \omega \mathrm{m}^*/(qk_\mathrm{F})$, $F_1 = f_1 N_0$ is the dimensionless Landau parameter and $N_0 = g\mathrm{m}^* k_\mathrm{F}/(2\pi^2)$ is the level density at the Fermi surface, with m^* being the effective mass.

To compare with the continued fraction approximation, we have to evaluate V_eff using the interaction (13). It is sufficient to write explicitly the first 3 terms of the expansion of V_eff and to obtain the complete series by recursion. The integrations involving G_HF have to be carried out in the Landau limit, i.e. $q = 0$, but finite ν. We get:

$$V_1 = f_0 + f_1\nu^2 \tag{15}$$

$$V_2 = -\frac{1}{3}\frac{f_1 F_1 \nu^2}{V_1 \chi_\mathrm{HF}} \tag{16}$$

$$V_3 = \frac{1}{9}\frac{f_1 F_1^2 \nu^2}{V_1 V_2 \chi_\mathrm{HF}^2} - V_2 \tag{17}$$

It is worth noticing that direct and exchange terms have been treated on the same footing in calculating the V_i's. Of course for $f_1 = 0$ only V_1 is needed and one gets the exact result. The effective interaction is

$$\begin{aligned} V_\mathrm{eff} &= V_1 + V_1 V_2 \chi_\mathrm{HF} + \left(V_1 V_2 V_3 + V_1 V_2^2\right)\chi_\mathrm{HF}^2 + \ldots \\ &= f_0 + f_1\nu^2 \left\{ 1 + \left(-\frac{1}{3}F_1\right) + \left(-\frac{1}{3}F_1\right)^2 + \ldots \right\} \\ &= f_0 + \frac{f_1\nu^2}{1 + \frac{1}{3}F_1} \ . \end{aligned} \tag{18}$$

One can see that this V_eff leads to the exact result (14) for the RPA response function.

III. RESULTS FOR A GOGNY INTERACTION

In this section we apply the continuous fraction method to calculate response functions in infinite symmetric matter for a realistic case, using the Gogny effective interaction D1 [3]. We choose this parametrization because at the mean field level there is a compensation between the direct and the density-dependent contributions. Thus, it may be expected that the relative contribution of the exchange term will be somehow enhanced. The purpose is to demonstrate the feasibility and rapid convergence of the method. We only present results at $T = 0$, for which the effects of the residual interaction are stronger.

The task of calculating the V_i's involves carrying out integrals over an increasing number of variables. We find convenient to use a multipole expansion of both the HF propagator G_HF and the p-h interaction V_ph, as we did in the numerically exact calculation of Ref. [12]:

$$\begin{aligned} G_\mathrm{HF}(q,\omega,\mathbf{k}_1) &= \sum_\ell G_\ell(q,\omega,k_1) Y_{\ell 0}(1) \ , \\ V_\mathrm{ph}(q,\mathbf{k}_1,\mathbf{k}_2) &= \sum_{\ell,m} v_\ell(q,k_1,k_2) Y^*_{\ell m}(1) Y_{\ell m}(2) \ . \end{aligned} \tag{19}$$

This allows to get rid of all integrations over angles and we are left with only integrals over the absolute values of momenta. For instance, we have

$$V_1 = \frac{g}{(\chi_\mathrm{HF})^2}\sum_\ell \langle G_\ell v_\ell G_\ell \rangle \ , \tag{20}$$

where the integrals implicit in the brackets refer now to the moduli k_i. Similar expressions can be obtained for other V_i's.

We can have an idea of the convergence rate by comparing the functions $V_1\chi_\mathrm{HF}$ and $V_2\chi_\mathrm{HF}$. This is shown in Fig. 1 for the case of a momentum transfer q=27 MeV. Notice that the scale used to plot $V_2\chi_\mathrm{HF}$ is about a factor of ten larger than that of $V_1\chi_\mathrm{HF}$. It can be seen that the imaginary parts of $V_2\chi_\mathrm{HF}$ are close to zero for the four spin-isospin channels. The real parts are generally small compared to 1, but the situation seems less favorable in the channel $(S,T) = (0,0)$. From the behavior shown in Fig.1 one can expect a rapid convergence of the calculated responses already at the level of V_2, although perhaps slower in the case of the $(0,0)$ channel.

We now examine the strength functions

$$S(q,\omega) = -\frac{1}{\pi}\mathrm{Im}\chi(q,\omega) \tag{21}$$

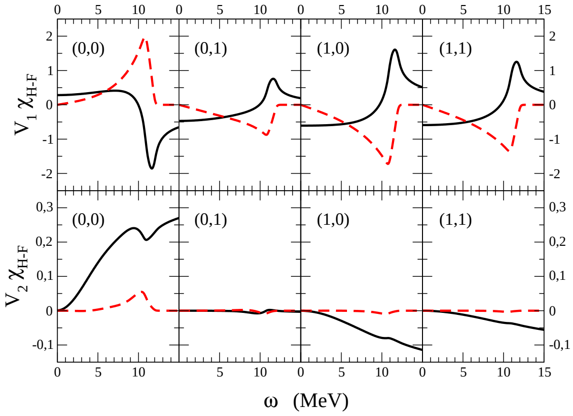

FIG. 1: (Color online) Real (solid line) and imaginary (dashed line) part of $V_1\chi_{\rm HF}$ (top row) and $V_2\chi_{\rm HF}$ (bottom row) for D1 interaction in nuclear matter at saturation density ρ_0. The transferred momentum is q=27 MeV. The (S,T) channels are shown in each panel.

obtained at various levels of approximation, as compared with the direct numerical solution of Eq. (3) presented in Ref. [12]. In Figs. 2-3 we show the RPA strength functions for two values of the momentum transfer, at about $k_{\rm F}/10$ and $k_{\rm F}$. The first order gives a reliable description of the strentgh function for all channels except $(0,0)$ as expected from the previous analysis. For the $(0,0)$ channel it is necessary to include the second order. Notice that the agreement is independent of the value of q, as no expansion in powers of q has been done. Indeed, as it can be seen in Eq. (11) the convergence of the approximation for the effective interaction does not rely on q but on the functions $V_i\chi_{\rm HF}$.

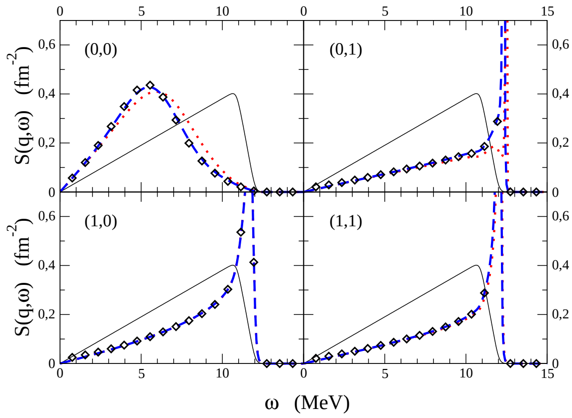

FIG. 2: (Color online) RPA Strength function (open diamonds) compared with continuous fraction approximation (1st order: dotted line, 2nd order: dashed line) calculated with Gogny D1 interaction in symmetric nuclear matter, at saturation density ρ_0 and momentum transfer q=27 MeV. The thin lines represent the uncorrelated HF strengths.

However, as the density increases the convergence is deteriorating. In Fig. 4 are plotted the strength functions $S^{(0,0)}$ and the functions $V_1\chi_{\rm HF}$, $V_2\chi_{\rm HF}$ at density $\rho = 2\rho_0$ in the $(0,0)$ spin-isospin channel. It can be seen that $V_2\chi_{\rm HF}$ is no longer small as compared to $V_1\chi_{\rm HF}$ and consequently, a reliable strength function should require at least the inclusion of third order terms in the effective interaction. On the other hand, for densities smaller than ρ_0 the approximation $V_{\rm eff} = V_1\chi_{\rm HF}$ is sufficient to get accurate results. Of course, the specific convergence found in each channel (S,T) depends on the specific interaction used.

Let us remind that the present approximation is not related to the relative importance of the direct and exchange contributions to the particle-hole interaction. Had the exchange term be small as compared to the direct one, a good approximation for the response function could be obtained by treating exactly the contribution of the latter and using some approximation for the contribution of the former term. This is not the case for the D1 interaction, as it can be seen in Table 1. The explicit expressions of these terms are given in Ref. [12]. As the exchange term depends on momenta k_1, k_2 and their relative angle, in the Table are plotted the monopole contributions at the Fermi surface $(k_1 = k_2 = k_{\rm F})$, for two values of q.

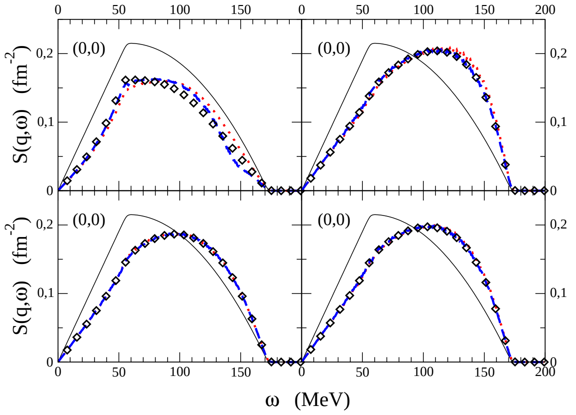

FIG. 3: (Color online) Same as Fig.2, for q=270 MeV.

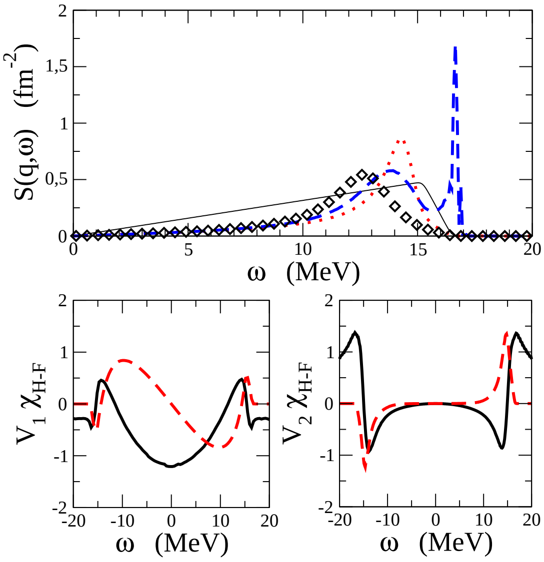

FIG. 4: (Color online) Same as Figs. 1 and 2 for the channel $(0, 0)$ and for $\rho = 2\rho_0$.

IV. CONCLUSIONS

We have examined the efficiency of the continuous fraction method for calculating RPA response functions in infinite nuclear matter using a typical finite range effective force. This issue originates from the need of having self-consistent theoretical predictions of nuclear responses calculated with realistic interactions.

We have found that, with the Gogny interaction D1 the continued fraction method is very efficient and the exact RPA response functions in the 4 spin-isospin channels are well reproduced already at first order. This is true when the nuclear density is of the order of, or less than the saturation density value. At higher densities it becomes necessary to include second and higher order terms. The rate of convergence is controlled by the decrease of the terms of successive orders $V_i\chi_{\rm HF}$. In our expansion the direct and exchange interactions are always treated on equal footing. This is important since in the nuclear case usually there occurs a strong cancellation of two large numbers, see Table I.

The encouraging results obtained in infinite nuclear matter open the way to important developments. For example, the continuous fraction method for response functions provides a simpler way to evaluate the propagation of neutrinos in dense matter such as inside neutron stars. The accuracy of results is under control by the rate of decrease of the successive terms $V_i\chi_{\rm HF}$. In finite nuclei, response functions can be calculated consistently with realistic effective interactions without diagonalizing RPA matrices of extremely large dimensions. This can be of some advantage for studying heavy and/or deformed nuclei.

(S,T) channel	(0,0)		(0,1)		(1,0)		(1,1)	
q (MeV)	27	135	27	135	27	135	27	135
$v^{(D)}_{\ell=0}(q)$	885	1129	-363	-459	845	798	-46	-146
$v^{(E)}_{\ell=0}(k_{1,2}=k_{\rm F})$	-1147	-1147	917	917	-420	-420	583	583

TABLE I: Direct (D) and exchange terms (E) in MeV.fm^{-3} of the D1 p-h interaction in nuclear matter, for $\rho=\rho_0$ and angular momentum ℓ=0.

Acknowledgments

This work is supported by the grant FIS2007-60133 (MEC, Spain) and by the IN2P3(France)-CICYT(Spain) exchange program.

[1] M.N. Harakeh and A. Van der Woude, *Giant Resonances* (Oxford University Press, 2001).
[2] D. Vautherin and D.M. Brink, Phys. Rev. **C5**, 626 (1972).
[3] D. Gogny, Proc. Int. Conf. Nuclear Self-consistent Fields, G. Ripka and M. Porneuf eds., North-Holland, Amsterdam (1975).
[4] C. García-Recio, J. Navarro, N. Van Giai and L.L. Salcedo, Ann. Phys. **214**, 293 (1992).
[5] J. Terasaki and J. Engel, Phys. Rev. **C 74**, 044301 (2006).
[6] K. Mizuyama, M. Matsuo and Y. Serizawa, arXiv:nucl-th/0706.1115
[7] J.P. Blaizot and D. Gogny, Nucl. Phys. **A 284**, 429 (1977).
[8] S. Péru, J.-F. Berger and P.F. Bortignon, Eur. Phys. J. **A 26**, 25 (2005).
[9] G.F. Berstch and S.F. Tsai, Phys. Rep. **18**C, 126 (1975).
[10] P. Schuck, R.W. Hasse, J. Jaenicke, C. Grégoire, B. Rémaud, F. Sébille and E. Suraud, Prog. Part. Nucl. Phys. **22**, 181 (1989).
[11] A. De Pace, Nucl. Phys. **A 635**, 163 (1998).
[12] J. Margueron, Nguyen Van Giai and J. Navarro, Phys. Rev. **C72**, 034311 (2005).
[13] A.L. Fetter and J.D. Walecka, *Quantum Theory of Many-Particle Systems*, McGraw-Hill (New York, 1971).