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Abstract

The aim of this article is to construct *à la* Perelomov and *à la* Barut–Girardello coherent states for a polynomial Weyl-Heisenberg algebra. This generalized Weyl-Heisenberg algebra, noted $\mathcal{A}_{\{\kappa\}}$, depends on r real parameters and is an extension of the $\mathcal{A}_\kappa$ one-parameter algebra (Daoud M and Kibler M R 2010 *J. Phys. A: Math. Theor.* **43** 115303) which covers the cases of the $su(1,1)$ algebra (for $\kappa > 0$), the $su(2)$ algebra (for $\kappa < 0$) and the h_4 ordinary Weyl-Heisenberg algebra (for $\kappa = 0$). For finite-dimensional representations of $\mathcal{A}_{\{\kappa\}}$ and $\mathcal{A}_{\{\kappa\},s}$, where $\mathcal{A}_{\{\kappa\},s}$ is a truncation of order s of $\mathcal{A}_{\{\kappa\}}$ in the sense of Pegg–Barnett, a connection is established with k-fermionic algebras (or quon algebras). This connection makes it possible to use generalized Grassmann variables for constructing certain coherent states. Coherent states of the Perelomov type are derived for infinite-dimensional representations of $\mathcal{A}_{\{\kappa\}}$ and for finite-dimensional representations of $\mathcal{A}_{\{\kappa\}}$ and $\mathcal{A}_{\{\kappa\},s}$ through a Fock–Bargmann analytical approach based on the use of complex (or bosonic) variables. The same approach is applied for deriving coherent states of the Barut–Girardello type in the case of infinite-dimensional representations of $\mathcal{A}_{\{\kappa\}}$. In contrast, the construction of *à la* Barut–Girardello coherent states for finite-dimensional representations of $\mathcal{A}_{\{\kappa\}}$ and $\mathcal{A}_{\{\kappa\},s}$ can be achieved solely at the price to replace complex variables by generalized Grassmann (or k-fermionic) variables. Some of the results are applied to $su(2)$, $su(1,1)$ and the harmonic oscillator (in a truncated or not truncated form).

1 Introduction

The coherent states play an essential role in many areas of physics (see [2, 33, 35, 42, 51, 59]). They were initially introduced by Schrödinger as quantum states whose dynamics is similar to a classical one [55]. There are three approaches to define coherent states for a quantum system described by a (Lie) group or algebra (see for instance [62]). The Perelomov approach defines coherent states by the action of group elements on a fiducial state of the representation space of a group [50, 51] (see also [36]). The Barut–Girardello approach defines coherent states as eigenstates of a lowering group generator [10]. The third approach is based on the minimization of Robertson-Schrödinger uncertainty relations for Hermitian generators of a group [56, 57] and leads to the so-called intelligent states [5, 6]. The three approaches generally yields different coherent states except for the harmonic oscillator for which they give equivalent results (the so-called Glauber states [37]).

In this paper, we shall be concerned with the construction of Perelomov and Barut–Girardello coherent states for a polynomial Weyl-Heisenberg algebra. It is known that the Perelomov approach can be used for any finite- or infinite-dimensional representation space of the considered group or algebra while the Barut–Girardello approach applies only to infinite-dimensional representation space [10, 50, 51] (see also [31]). It was one of the object of this work to find a way to define Barut–Girardello coherent states for a finite-dimensional representation space.

The original strategy adopted here is as follows. For Perelomov coherent states, we work with a Fock–Bargmann space for which the annihilation operator of the polynomial Weyl-Heisenberg algebra acts as derivative by a complex variable both in the finite- and infinite-dimensional cases. For Barut–Girardello coherent states, the creation operator acts on the Fock–Bargmann space as multiplication by a complex variable in the infinite-dimensional case and as multiplication by a generalized Grassmann variable in the finite-dimensional case.

The paper is organized as follows. Section 2 deals with a polynomial Weyl-Heisenberg algebra, a r-parameter algebra denoted $\mathcal{A}_{\{\kappa\}}$ with $\{\kappa\} \equiv \{\kappa_1, \kappa_2, \cdots, \kappa_r\}$, which generalizes the $\mathcal{A}_\kappa$ one-parameter oscillator algebra worked out in [8, 22]. Such an algebra is shown to have finite- or infinite-dimensional representations depending on the values of the $\{\kappa_1, \kappa_2, \cdots, \kappa_r\}$ parameters. In the case of an infinite-dimensional representation, it is shown how to restrict $\mathcal{A}_{\{\kappa\}}$ to a $\mathcal{A}_{\{\kappa\},s}$ truncated algebra with a representation of finite dimension s. In order to define Barut–Girardello coherent states in the finite-dimensional case, Section 3 is devoted to a review of k-fermion operators (which generalize ordinary fermion operators) and their realization in terms of generalized Grassmann variables. A connection between a k-fermionic algebra and either $\mathcal{A}_{\{\kappa\}}$ with a finite-dimensional representation or $\mathcal{A}_{\{\kappa\},s}$ is studied in Section 4. Section 5 and 6 deal with the derivation of Perelomov and Barut–Girardello coherent states for $\mathcal{A}_{\{\kappa\}}$ and $\mathcal{A}_{\{\kappa\},s}$. At this point, a word of explanation of the title is in order. The Perelomov states derived in the present paper in finite and infinite dimensions are of bosonic type (i.e., labeled by a complex variable) while the Barut-Girardello states are of k-fermionic type (i.e. labeled by a Grassmann variable) in finite dimension and of bosonic type in infinite dimension.

2 Generalized Weyl-Heisenberg algebra

In the last two decades, due to the development of the theory of quantum algebras, non-linear extensions of the usual Weyl-Heisenberg algebra (or oscillator algebra) attracted a lot of attention. Many variants of generalized Weyl-Heisenberg algebras were proposed and studied from different viewpoints (e.g., see [24, 38, 53]). All generalized oscillators can be unified into a common framework by taking the product of the creation and annihilation operators (which is the N number operator for the usual oscillator) to be $F(N)$, where the F structure function characterizes the generalization scheme. Different structure functions correspond to different deformed oscillators. Of particular interest are the so-called polynomial Weyl-Heisenberg algebras in which $F(N)$ is a polynomial in N and consequently the commutator of the annihilation and creation operators becomes a polynomial in N too. From a physical point of view, such polynomial Weyl-Heisenberg algebras offer the advantage of dealing with quantum systems with non-linear discrete spectrum [17, 22, 27, 29].

2.1 The $\mathcal{A}_{\{\kappa\}}$ algebra

We start with the generalized Weyl-Heisenberg algebra on $\mathbb{C}$ spanned by the linear operators a^- (annihilation operator), a^+ (creation operator) and N (number operator) satisfying the commutation relations

$$[a^-, a^+] = G(N) \qquad [N, a^-] = -a^- \qquad [N, a^+] = +a^+ \tag{1}$$

and the Hermitian conjugation conditions

$$a^+ = (a^-)^\dagger \qquad N = N^\dagger. \tag{2}$$

The G function in (1) is such that

$$G(N) = (G(N))^\dagger. \tag{3}$$

Of course, the case $G(N) = I$, where I is the identity operator, corresponds to the usual Weyl-Heisenberg algebra or harmonic oscillator algebra. Various realizations of G are known in the literature [17, 20, 21, 22, 24, 29, 32, 38, 48, 53]. In the present paper, we shall be concerned with a class of polynomial Weyl-Heisenberg algebras characterized by

$$G(N) = F(N + I) - F(N) \tag{4}$$

with the F function defined by

$$F(N) = N[I + \kappa_1(N - I)][I + \kappa_2(N - I)] \cdots [I + \kappa_r(N - I)] \tag{5}$$

where the κ_i's ($i = 1, 2, \cdots, r$) are real parameters (for instance, see [32]). We note $\mathcal{A}_{\{\kappa\}}$, with $\{\kappa\} \equiv \{\kappa_1, \kappa_2, \cdots, \kappa_r\}$, the generalized Weyl-Heisenberg algebra (or generalized oscillator algebra) defined via (1)-(5).

The $F(N)$ polynomial of order $r + 1$ with respect to N can be developed as

$$F(N) = N \sum_{i=0}^{r} s_i (N - I)^i \tag{6}$$

in terms of the coefficients (totally symmetric under permutation group S_r)

$$s_0 = 1 \qquad s_i = \sum_{j_1<j_2<\cdots<j_i} \kappa_{j_1}\kappa_{j_2}\cdots\kappa_{j_i} \quad (i = 1, 2, \cdots, r) \tag{7}$$

where the indices $j_1, j_2, \cdots, j_i$ take the values $1, 2, \cdots, r$. Then, the $G(N)$ operator can be written

$$G(N) = I + \sum_{i=1}^{r} s_i \left[(N + I)N^i - N(N - I)^i \right] \tag{8}$$

which clearly indicates that $\mathcal{A}_{\{\kappa\}}$ with $\{\kappa\} \equiv \{0, 0, \cdots, 0\}$ coincides with the usual Weyl-Heisenberg algebra.

The $\mathcal{A}_{\{\kappa\}}$ r-parameter algebra covers the cases of (i) the extended harmonic oscillator algebra [53], (ii) the fractional oscillator algebra [20], and (iii) the W_k algebra introduced in the context of fractional supersymmetric quantum mechanics of order k [21, 40]. As a particular case, algebra $\mathcal{A}_{\{\kappa\}}$ with $\kappa_1 = \kappa$ and $r = 1$ is nothing but the $\mathcal{A}_\kappa$ algebra worked out in [22] and corresponding to

$$G(N) = I + 2\kappa N. \tag{9}$$

Algebra $\mathcal{A}_\kappa$ defined by (1), (2) and (9) turns out to be of particular interest when dealing with dynamical symmetries of some exactly solvable quantum systems. More precisely, $\mathcal{A}_{\kappa=0}$ corresponds to the usual oscillator system while $\mathcal{A}_{\kappa<0}$ and $\mathcal{A}_{\kappa>0}$ are relevant to the Morse and Pöschl-Teller systems, respectively [21, 22]. Note also that the $\mathcal{A}_\kappa$ one-parameter algebra provides a unified scheme to deal with the $su(2)$ algebra (for $\kappa < 0$), the $su(1,1)$ algebra (for $\kappa > 0$), and the usual Weyl-Heisenberg algebra (for $\kappa = 0$) [8, 22]. More generally, the $\mathcal{A}_{\{\kappa\}}$ algebra can be viewed as a special class of the polynomial extensions of $su(2)$ and $su(1,1)$ discussed in [15] and [58], respectively.

2.2 The $\mathcal{F}_{\{\kappa\}}$ representation space

Let $\mathcal{F}_{\{\kappa\}}$ be the Fock-Hilbert space on which operators a^-, a^+ and N act. This space is spanned by a complete set $\{|n\rangle \ : \ n = 0, 1, 2, \cdots\}$ of orthonormal states which are eigenstates of operator N. It can be formally verified that the relations

$$a^-|n\rangle = \sqrt{F(n)}e^{+i[F(n)-F(n-1)]\varphi}|n - 1\rangle \tag{10}$$

$$a^+|n\rangle = \sqrt{F(n + 1)}e^{-i[F(n+1)-F(n)]\varphi}|n + 1\rangle \tag{11}$$

$$a^-|0\rangle = 0 \qquad N|n\rangle = n|n\rangle \tag{12}$$

define an Hilbertian representation of $\mathcal{A}_{\{\kappa\}}$. In Eqs. (10) and (11), φ is an arbitrary real parameter and the F structure function satisfies the conditions

$$F(0) = 0 \qquad F(n) > 0 \quad (n = 1, 2, 3, \cdots) \tag{13}$$

and

$$F(n + 1) - F(n) = G(n) \iff F(n) = \sum_{m=0}^{n-1} G(m). \tag{14}$$

The dimension of the $\mathcal{F}_{\{\kappa\}}$ space can be finite or infinite. It is determined by condition (13) which in turn depends on parameters κ_i $(i = 1, 2, \cdots, r)$. In the rest of this paper, we shall assume that

$$\kappa_1 \in \mathbb{R} \qquad \kappa_i \in \mathbb{R}_+ \quad (i = 2, 3, \cdots, r). \tag{15}$$

Hence, we have

$$F(n) > 0 \ \Rightarrow \ 1 + \kappa_1(n-1) > 0 \quad (n \neq 0) \tag{16}$$

so that the dimension of the considered representation of $\mathcal{A}_{\{\kappa\}}$ is completely fixed by the sign of κ_1 in the following way.

- For $\kappa_1 \geq 0$, the $\mathcal{F}_{\{\kappa\}}$ space is infinite-dimensional. In this case, we have

$$F(n) = n \prod_{i=1}^{r} [1 + \kappa_i(n-1)] \tag{17}$$

  which gives $F(n) = n$ for the harmonic oscillator.

- For $\kappa_1 < 0$, there exists a finite number of states satisfying condition (16). In this case, n can take the values

$$n = 0, 1, \cdots, E(-\frac{1}{\kappa_1}) \tag{18}$$

  where we use $E(x)$ to denote the integer part of x. From now on, we shall assume that $-1/\kappa_1 \in \mathbb{N}^*$ when $\kappa_1 < 0$. Therefore, the dimension of $\mathcal{F}_{\{\kappa\}}$ with $\kappa_1 < 0$ is

$$d = 1 - \frac{1}{\kappa_1} \qquad d \in \mathbb{N} \setminus \{0, 1\}. \tag{19}$$

  Note that Eq. (19) easily follows from $\operatorname{Tr} G(N) = 0$ that holds for $\kappa_1 < 0$ and $\kappa_i \geq 0$ $(i = 2, 3, \cdots, r)$. In this case, we have

$$F(n) = n \frac{d-n}{d-1} \prod_{i=2}^{r} [1 + \kappa_i(n-1)] \tag{20}$$

  which gives $F(n) = n(d-n)/(d-1)$ for $r = 1$ like in [22].

The finiteness of the $\mathcal{F}_{\{\kappa\}}$ space for $\kappa_1 < 0$ and $\kappa_i \geq 0$ $(i = 2, 3, \cdots, r)$ induces properties of a^- and a^+ which differ from those corresponding to an infinite-dimensional space. Along this line, we have the limit condition

$$a^+|d-1\rangle = 0 \tag{21}$$

and the nilpotency relations

$$(a^-)^d = (a^+)^d = 0 \tag{22}$$

within the $\mathcal{F}_{\{\kappa\}}$ space generated by $\{|0\rangle, |1\rangle, \cdots, |d-1\rangle\}$.

2.3 The H Hamiltonian

Going back to the general case where $\kappa_1 \in \mathbb{R}$ and $\kappa_i \in \mathbb{R}_+$ $(i = 2, 3, \cdots, r)$, we have

$$a^+ a^- |n\rangle = F(n)|n\rangle. \tag{23}$$

We can thus introduce the operator

$$H \equiv F(N) = a^+ a^- \tag{24}$$

which generalizes the Hamiltonian for the one-dimensional harmonic oscillator up to an additive constant. We refer H to as the Hamiltonian associated with generalized oscillator algebra $\mathcal{A}_{\{\kappa\}}$. Note that $F(N) = N$ for the usual harmonic oscillator which corresponds to $\kappa_i = 0$ $(i = 1, 2, \cdots, r)$. For $\kappa_1 \in \mathbb{R}^*$ and $\kappa_i = 0$ $(i = 2, 3, \cdots, r)$, the H Hamiltonian is quadratic in N and can be related to some exactly solvable quantum systems as the Morse and Pöschl-Teller systems [22]. For higher order in N, H can be used to describe some non linear effects as for instance the isotopic effect in vibrational spectra of diatomic molecules (cf. [12]). It is also relevant for dealing with higher-order supersymmetric harmonic oscillators [29]. From a general point of view, it is interesting to note that energy gap $F(n+1) - F(n)$ between two consecutive states behaves like n^r; therefore, for an infinite-dimensional space, we can ignore, in a perturbative scheme, the highly excited states with n large and thus work with a subspace of finite dimension.

2.4 The $\mathcal{A}_{\{\kappa\},s}$ truncated algebra

As discussed above, in the case where $\kappa_i \geq 0$ $(i = 1, 2, \cdots, r)$, the $\mathcal{F}_{\{\kappa\}}$ representation space of the $\mathcal{A}_{\{\kappa\}}$ algebra is infinite-dimensional. In this case, to restrict $\mathcal{A}_{\{\kappa\}}$ to a *truncated* algebra acting on a subspace of $\mathcal{F}_{\{\kappa\}}$, we introduce operators $a^-(s)$ and $a^+(s)$

$$a^+(s) = a^+ - \sum_{n=s}^{\infty} \sqrt{F(n)} e^{-i[F(n)-F(n-1)]\varphi} |n\rangle\langle n-1| \tag{25}$$

$$a^-(s) = a^- - \sum_{n=s}^{\infty} \sqrt{F(n)} e^{+i[F(n)-F(n-1)]\varphi} |n-1\rangle\langle n| \tag{26}$$

(we follow here the usual tradition in speaking of truncated algebra but it should be realized that the truncation applies to the representation space). They satisfy the following commutation relations

$$[a^-(s), a^+(s)] = G_s(N) - F(s)|s-1\rangle\langle s-1| \qquad [N, a^\pm(s)] = \pm a^\pm(s) \tag{27}$$

where

$$G_s(N) = \sum_{n=0}^{s-1} [F(n+1) - F(n)] |n\rangle\langle n| \tag{28}$$

and thus span with N an algebra that we denote $\mathcal{A}_{\{\kappa\},s}$. The $\mathcal{A}_{\{\kappa\},s}$ algebra is refered to as truncated generalized oscillator algebra of order s. It generalizes algebra $\mathcal{A}_{\kappa,s}$ defined in [22] which admits as a particular case the truncated Weyl-Heisenberg algebra introduced by Pegg end Barnett [49]. Indeed, the $\mathcal{A}_{\kappa,s}$ algebra with $\kappa = 0$ is identical to the Pegg–Barnett oscillator algebra.

It is easy to check that

$$a^-(s)|n\rangle = \sqrt{F(n)}e^{+i[F(n)-F(n-1)]\varphi}|n-1\rangle \tag{29}$$

$$a^+(s)|n\rangle = \sqrt{F(n+1)}e^{-i[F(n+1)-F(n)]\varphi}|n+1\rangle \tag{30}$$

$$a^-(s)|0\rangle = 0 \qquad a^+(s)|s-1\rangle = 0 \qquad N|n\rangle = n|n\rangle \tag{31}$$

for $n = 0, 1, \cdots, s-1$. It follows that Eqs. (29)-(31) define a representation of $\mathcal{A}_{\{\kappa\},s}$ on the $\mathcal{F}_{\{\kappa\},s}$ space generated by the $\{|0\rangle, |1\rangle, \cdots, |s-1\rangle\}$ set. In this s-dimensional representation, operators $a^-(s)$ and $a^+(s)$ satisfy

$$[a^-(s)]^s = [a^+(s)]^s = 0 \tag{32}$$

to be compared with (22). It should be noted that Eqs. (22) and (32) are identical for $d = s = k \in \mathbb{N} \setminus \{0, 1\}$ to the nilpotency relations describing the so-called k-fermions that are objects interpolating between fermions (for $k = 2$) and bosons (for $k \to \infty$) [18, 19].

3 Introducing k-fermions

3.1 The k-fermionic algebra

We first define a quon algebra. The A_q quon algebra is generated by an annihilation operator (f_-), a creation operator (f_+) and a number operator (N) with the relations [18, 19, 39]

$$f_-f_+ - qf_+f_- = I \qquad [N, f_-] = -f_- \qquad [N, f_+] = +f_+ \qquad N = N^\dagger \tag{33}$$

and

$$(f_-)^k = (f_+)^k = 0 \tag{34}$$

where q is primitive kth root of unity:

$$q = \exp\left(\frac{2\pi i}{k}\right) \qquad k \in \mathbb{N} \setminus \{0, 1\}. \tag{35}$$

Note that the q-commutation relation in Eq. (33) is satisfied by

$$f_-f_+ = [N+I]_q \qquad f_+f_- = [N]_q \tag{36}$$

with

$$[X]_q = \frac{1-q^X}{1-q} \tag{37}$$

where X may be an operator or a complex number.

It is important to realize that quon algebra A_q differs from that introduced by Arik and Coon [7], Biedenharn [14] and Macfarlane [45] because q is not here a real number and operators f_- and f_+ satisfy nilpotency relations (of order k). Obviously, Eq. (33) shows that f_- and f_+ cannot be connected by Hermitian conjugation (i.e., $f_+ \neq (f_-)^\dagger$) when q is a root of unity except for $k = 2$ (i.e., $q = -1$) or $k \to \infty$ (i.e., $q = +1$). The A_{-1} algebra corresponds to ordinary fermion operators with

$(f_+)^2 = (f_-)^2 = 0$, a relation that reflects the Pauli exclusion principle. In the limiting case where $k \to \infty$, the A_{+1} algebra corresponds to ordinary boson operators; in this case, Eq. (33) describes the usual harmonic oscillator algebra. For q a primitive k-th root of unity different from ± 1, operators f_- and f_+ are referred to as k-fermion operators.

It is a simple exercise to check that the actions

$$f_-|n\rangle = \left([n]_q\right)^{\frac{1}{2}} |n-1\rangle \qquad f_-|0\rangle = 0 \tag{38}$$

$$f_+|n\rangle = \left([n+1]_q\right)^{\frac{1}{2}} |n+1\rangle \qquad f_+|k-1\rangle = 0 \tag{39}$$

and

$$N|n\rangle = n|n\rangle \tag{40}$$

on a Fock-Hilbert space of dimension k, noted F_q and isomorphic with $\mathcal{F}_{\{\kappa\}}$ with $d = k$, defines a k-dimensional representation of A_q. Of course, $\dim F_{-1} = 2$ for fermions while $\dim F_{+1}$ is infinite for bosons. Any vector of F_q can be obtained by repeated application of f_+ on the *ground* state:

$$|n\rangle = \frac{1}{([n]_q!)^{\frac{1}{2}}}(f_+)^n|0\rangle \quad (n = 0, 1, \cdots, k-1) \tag{41}$$

where the $[n]_q$-factorial is defined by

$$[0]_q! = 1 \qquad [n]_q! = \prod_{i=1}^{n}[i]_q \quad n \in \mathbb{N} \setminus \{0\} \tag{42}$$

as is usual.

It was mentioned above that the relation $f_+ = (f_-)^\dagger$ does not hold in general. Thus, we introduce the operators

$$f_+^+ = (f_+)^\dagger \qquad f_-^+ = (f_-)^\dagger. \tag{43}$$

The three operators f_+^+, f_-^+ and N span the $A_{\bar{q}}$ quon algebra defined by

$$f_+^+ f_-^+ - \bar{q} f_-^+ f_+^+ = I \qquad [N, f_+^+] = -f_+^+ \qquad [N, f_-^+] = +f_-^+ \qquad N = N^\dagger \tag{44}$$

and

$$(f_+^+)^k = (f_-^+)^k = 0. \tag{45}$$

Operator f_+^+ plays the role of an annihilation operator and operator f_-^+ the one of a creation operator for the $A_{\bar{q}}$ algebra. We can check that f_+^+ and f_-^+ act on F_q according to

$$f_+^+|n\rangle = \left([n]_{\bar{q}}\right)^{\frac{1}{2}} |n-1\rangle \qquad f_+^+|0\rangle = 0 \tag{46}$$

$$f_-^+|n\rangle = \left([n+1]_{\bar{q}}\right)^{\frac{1}{2}} |n+1\rangle \qquad f_-^+|k-1\rangle = 0. \tag{47}$$

At this point, it is worth noticing that the relation

$$f_- f_+^+ = q^{-\frac{1}{2}} f_+^+ f_- \iff f_+ f_-^+ = q^{+\frac{1}{2}} f_-^+ f_+ \tag{48}$$

holds on the F_q space. We shall refer to k-fermionic algebra, noted Σ_q, the algebra generated by the (f_+, f_-^+) and (f_-, f_+^+) pairs together with the N operator satisfying Eqs. (33), (34), (44), (45) and (48).

3.2 Grassmannian realization

Equations (34) and (45) suggest that we look for a realization of the (f_+, f_-^+) and (f_-, f_+^+) pairs in terms of generalized Grassmann variables $(\theta, \bar{\theta})$ and their q- and $\bar{q}$-derivatives $(\partial_\theta, \partial_{\bar{\theta}})$. Generalized Grassmann variables θ and $\bar{\theta}$ of order k, introduced in connection with quantum groups and fractional supersymmetry, satisfy

$$\theta^k = \bar{\theta}^k = 0 \tag{49}$$

(see [1, 25, 30, 44, 46, 54]). The sets $\{I, \theta, \cdots, \theta^{k-1}\}$ and $\{I, \bar{\theta}, \cdots, \bar{\theta}^{k-1}\}$ span isomorphic Grassmann algebras. The q- and $\bar{q}$-derivatives are formally defined by

$$\partial_\theta f(\theta) = \frac{f(q\theta) - f(\theta)}{(q-1)\theta} \qquad \partial_{\bar{\theta}} g(\bar{\theta}) = \frac{g(\bar{q}\bar{\theta}) - g(\bar{\theta})}{(\bar{q}-1)\bar{\theta}} \tag{50}$$

where $f : \theta \mapsto f(\theta)$ and $g : \bar{\theta} \mapsto g(\bar{\theta})$ are arbitrary functions. The ∂_θ and $\partial_{\bar{\theta}}$ operators satisfy

$$\partial_\theta \theta^n = [n]_q \, \theta^{n-1} \qquad \partial_{\bar{\theta}} \bar{\theta}^n = [n]_{\bar{q}} \, \bar{\theta}^{n-1} \tag{51}$$

for $n = 0, 1, \cdots, k-1$. Hence, for functions f and g such that

$$f(\theta) = \sum_{n=0}^{k-1} a_n \theta^n \qquad g(\bar{\theta}) = \sum_{n=0}^{k-1} b_n \bar{\theta}^n \tag{52}$$

where the a_n and b_n coefficients in the expansions are complex numbers, we easily show that

$$(\partial_\theta)^k f(\theta) = (\partial_{\bar{\theta}})^k g(\bar{\theta}) = 0. \tag{53}$$

As a consequence, we assume that the conditions

$$(\partial_\theta)^k = (\partial_{\bar{\theta}})^k = 0 \tag{54}$$

hold in addition to (49). We can also define an integration process. Following Majid and Rodríguez-Plaza [46], we take

$$\int \theta^n d\theta = \int \bar{\theta}^n d\bar{\theta} = 0 \quad (n = 0, 1, \cdots, k-2) \tag{55}$$

and

$$\int \theta^{k-1} d\theta = \int \bar{\theta}^{k-1} d\bar{\theta} = 1 \tag{56}$$

which gives the Berezin integration for the $k = 2$ particular case ($k = 2$ corresponds to the ordinary Grassmann variables used in supersymmetry).

As a result, we can show that the correspondences

$$f_- \longrightarrow \partial_\theta \qquad f_+ \longrightarrow \theta \qquad f_+^+ \longrightarrow \partial_{\bar{\theta}} \qquad f_-^+ \longrightarrow \bar{\theta} \tag{57}$$

provide us with a realization of the q- and $\bar{q}$-commutators in (33) and (44) and of the nilpotency relations in (34) and (45). Note that (48) becomes

$$\theta\bar{\theta} = q^{+\frac{1}{2}} \bar{\theta}\theta \qquad \partial_\theta \partial_{\bar{\theta}} = q^{-\frac{1}{2}} \partial_{\bar{\theta}} \partial_\theta \tag{58}$$

in the realization afforded by (57).

4 From generalized Weyl-Heisenberg algebra to quon algebras

4.1 Obtaining the A_q algebra

We may ask how to pass from the $\mathcal{A}_{\{\kappa\}}$ generalized Weyl-Heisenberg algebra to the A_q quon algebra. Since A_q admits a representation of dimension k ($q = \exp(2\pi i/k)$), we shall consider in this section algebra $\mathcal{A}_{\{\kappa\}}$ with $\{\kappa\}$ given by

$$\kappa_1 = -\frac{1}{k-1} < 0 \qquad \kappa_i \geq 0 \quad (i = 2, 3, \cdots, r) \tag{59}$$

with $k \in \mathbb{N} \setminus \{0, 1\}$. Our aim is to find three operators, say A_-, A_+ and N_A, acting on $\mathcal{F}_{\{\kappa\}}$ and expressed as functions of generators a_-, a_+ and N of $\mathcal{A}_{\{\kappa\}}$, such that A_-, A_+ and N_A span A_q. For this purpose, we put

$$A_- = \sum_{i=1}^{k-1} C_i (a^+)^{i-1} (a^-)^i \qquad A_+ = a^+ \qquad N_A = N \tag{60}$$

where the C_i's are expansion coefficients to be determined.

As a first step, in order to determine the C_i coefficients, it is sufficient to require that

$$A_+ A_- = [N]_q \tag{61}$$

on the $\mathcal{F}_{\{\kappa\}}$ representation space (cf. Eq. (36)). This yields

$$\sum_{i=1}^{k-1} C_i (a^+)^i (a^-)^i |n\rangle = [n]_q |n\rangle. \tag{62}$$

A simple development of the lhs of (62) leads to the following sytem of $k-1$ unknowns with $k-1$ equations:

$$\sum_{i=1}^{n} C_i \frac{F(n)!}{F(n-i)!} = [n]_q \quad (n = 1, 2, \cdots, k-1) \tag{63}$$

where the $F(n)$-factorial is defined by

$$F(0)! = 1 \qquad F(n)! = \prod_{i=1}^{n} F(i) \quad n \in \mathbb{N} \setminus \{0\}. \tag{64}$$

The $F(i)$'s occurring in (63) are taken from (20) with $d = k$. System (63) can be written as

$$T \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_{k-1} \end{pmatrix} = \begin{pmatrix} 1 \\ [2]_q \\ \vdots \\ [k-1]_q \end{pmatrix} \tag{65}$$

where $(k-1) \times (k-1)$-matrix T reads

$$T = \begin{pmatrix} F(1)! & 0 & 0 & \cdots & 0 \\ F(2) & F(2)! & 0 & \cdots & 0 \\ F(3) & F(3)F(2) & F(3)! & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F(k-1) & F(k-1)F(k-2) & F(k-1)F(k-2)F(k-3) & \cdots & F(k)! \end{pmatrix} \tag{66}$$

Clearly

$$\det T = F(1)!F(2)! \cdots F(k-1)! \neq 0 \tag{67}$$

so that there is a unique solution for the C_i coefficients which can be calculated from

$$\begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_{k-1} \end{pmatrix} = T^{-1} \begin{pmatrix} 1 \\ [2]_q \\ \vdots \\ [k-1]_q \end{pmatrix} \tag{68}$$

by inverting (65).

As a second step, we prove that

$$A_- A_+ = [N + I]_q \tag{69}$$

holds for the operators A_- and A_+ given by (60) with the C_i coefficients derived from (63). This can be seen as follows. We have

$$(a^+)^{i-1}(a^-)^i a^+|n\rangle = \begin{cases} F(n+1)\frac{F(n)!}{F(n+1-i)!}|n\rangle \quad \text{if} \quad i \leq n+1 \\ \\ 0 \quad \text{if} \quad i > n+1. \end{cases} \tag{70}$$

This implies

$$A_- A_+|n\rangle = \sum_{i=1}^{n+1} C_i \frac{F(n+1)!}{F(n+1-i)!}|n\rangle. \tag{71}$$

By using (63), Eq. (71) can be rewritten as

$$A_- A_+|n\rangle = [n+1]_q|n\rangle \tag{72}$$

which shows that (69) is satisfied on $\mathcal{F}_{\{\kappa\}}$.

The two preceding steps leads to

$$A_- A_+ - qA_+ A_- = I. \tag{73}$$

Furthermore, it is trivial to check that

$$[N, A_\pm] = \pm A_\pm \qquad (A_-)^k = (A_+)^k = 0. \tag{74}$$

As a result, Eqs. (73) and (74) correspond to Eqs. (33) and (34). Therefore, A_-, A_+ and N_A generate quon algebra A_q with $q = \exp(2\pi i/k)$.

We close this section with the action of annihilation operator A_- and creation operator A_+ on $\mathcal{F}_{\{\kappa\}}$:

$$A_-|n\rangle = \frac{[n]_q}{\sqrt{F(n)}} e^{+i[F(n)-F(n-1)]\varphi}|n-1\rangle \qquad A_-|0\rangle = 0 \tag{75}$$

$$A_+|n\rangle = \sqrt{F(n+1)} e^{-i[F(n+1)-F(n)]\varphi}|n+1\rangle \qquad A_+|k-1\rangle = 0. \tag{76}$$

The derivation of (76) is immediate while the proof of (75) requires the use of (63).

4.2 Obtaining the $A_{\bar{q}}$ quon algebra

A similar connection between $\mathcal{A}_{\{\kappa\}}$ and $A_{\bar{q}}$ can be obtained. By defining

$$A_{+}^{+} = a^{-} \qquad A_{-}^{+} = \sum_{i=1}^{k-1} \overline{C}_i (a^{+})^i (a^{-})^{i-1} \qquad N_A = N \tag{77}$$

it straightforwardly follows that A_{+}^{+}, A_{-}^{+} and N_A span quon algebra $A_{\bar{q}}$. The action of A_{+}^{+} and A_{-}^{+} on $\mathcal{F}_{\{\kappa\}}$ is given by

$$A_{+}^{+}|n\rangle = \sqrt{F(n)} e^{+i[F(n)-F(n-1)]\varphi} |n-1\rangle \qquad A_{+}^{+}|0\rangle = 0 \tag{78}$$

$$A_{-}^{+}|n\rangle = \frac{[n+1]_{\bar{q}}}{\sqrt{F(n+1)}} e^{-i[F(n+1)-F(n)]\varphi} |n+1\rangle \qquad A_{-}^{+}|k-1\rangle = 0 \tag{79}$$

cf. (75) and (76).

4.3 Passage formulas

The main result of Sections 6.1 and 6.2 is that triplets $\{A_-, A_+, N\}$ and $\{A_{+}^{+}, A_{-}^{+}, N\}$ span the A_q and $A_{\bar{q}}$ quon algebras, respectively. This does not mean that operators $A_{\pm}$ and $A_{\pm}^{+}$ can be identified with fermion operators $f_{\pm}$ and $f_{\pm}^{+}$, respectively. Indeed, it can be verified that the following passage formulas

$$A_- = \left(\frac{[N+I]_q}{F(N+I)} \right)^{\frac{1}{2}} e^{+iG(N)\varphi} f_- \qquad A_+ = f_+ \left(\frac{F(N+I)}{[N+I]_q} \right)^{\frac{1}{2}} e^{-iG(N)\varphi} \tag{80}$$

and

$$A_{+}^{+} = \left(\frac{F(N+I)}{[N+I]_{\bar{q}}} \right)^{\frac{1}{2}} e^{+iG(N)\varphi} f_{+}^{+} \qquad A_{-}^{+} = f_{-}^{+} \left(\frac{[N+I]_{\bar{q}}}{F(N+I)} \right)^{\frac{1}{2}} e^{-iG(N)\varphi} \tag{81}$$

are compatible with the action of operators $f_{\pm}$, $f_{\pm}^{+}$, $A_{\pm}$ and $A_{\pm}^{+}$ on the $\mathcal{F}_{\{\kappa\}}$.

Finally, note that the analysis presented in Section 4 can be extended to truncated Weyl-Heisenberg algebra $\mathcal{A}_{\{\kappa\},s}$ modulo some obvious changes.

5 *à la* Perelomov coherent states

5.1 Infinite-dimensional Fock-Hilbert space

For $\kappa_i \geq 0$ ($i = 1, 2, \cdots, r$), let us look for φ-dependent coherent states associated with the $\mathcal{A}_{\{\kappa\}}$ algebra in the form

$$|z, \varphi\rangle = \sum_{n=0}^{\infty} \overline{a_n} z^n |n\rangle \tag{82}$$

where z is a complex variable and the a_n coefficients depend on φ. Inspired by Bargmann [9], we assume that the basis vectors of the $\mathcal{F}_{\{\kappa\}}$ infinite-dimensional space are realized as follows

$$|n\rangle \longrightarrow a_n z^n \equiv \langle \bar{z}, \varphi | n \rangle \tag{83}$$

and the a^- annihilation operator acts on $\mathcal{F}_{\{\kappa\}}$ as a derivation according to

$$a^- \longrightarrow \frac{d}{dz}. \tag{84}$$

Then, the application of the correspondence rules (83) and (84) to the expression of $a^-|n\rangle$ given by (10) yields the following recursion relation

$$na_n = \sqrt{F(n)}e^{+i[F(n)-F(n-1)]\varphi}a_{n-1} \tag{85}$$

which leads to

$$a_n = a_0\frac{\sqrt{F(n)!}}{n!}e^{+iF(n)\varphi} \tag{86}$$

where $F(n)!$ follows from (64) and (17). As a result, we get the normalized state vectors

$$|z,\varphi\rangle = \mathcal{N}^{-1}\sum_{n=0}^{\infty}\frac{\sqrt{F(n)!}}{n!}z^n e^{-iF(n)\varphi}|n\rangle \tag{87}$$

where the $\mathcal{N}$ normalization factor is formally given by

$$|\mathcal{N}|^2 = \sum_{n=0}^{\infty}\frac{F(n)!}{(n!)^2}|z|^{2n} \tag{88}$$

subject to convergence. It should be observed that the $|z,\varphi\rangle$ vectors given by (87) and (88) with $\kappa_i = 0$ $(i = 1, 2, \cdots, r)$ and $\varphi = 0$ are coherent states for the usual harmonic oscillator. The series in (88) diverges for $r \geq 2$ when $\kappa_i \neq 0$ $(i = 1, 2, \cdots, r)$. For $r = 1$, it converges in the disk

$$\mathcal{D} = \{z \in \mathbb{C} \ : \ |z| < \frac{1}{\sqrt{\kappa_1}}\}. \tag{89}$$

Therefore, only the $r = 1$ case deserves to be considered here (in contrast with Section 6.1).

For $r = 1$, let us consider the special case

$$\kappa_1 \equiv \kappa = \frac{1}{\ell} \qquad k \in \mathbb{N}^*. \tag{90}$$

In this case, Eqs. (83) and (84) can be completed by

$$N \longrightarrow z\frac{d}{dz} \qquad a^+ \longrightarrow z\left(1 + \frac{1}{\ell}z\frac{d}{dz}\right). \tag{91}$$

The $|z,\varphi\rangle$ state vectors for the corresponding $\mathcal{A}_\kappa$ algebra follow from (87)-(90). We obtain

$$|z,\varphi\rangle = \mathcal{N}^{-1}\sum_{n=0}^{\infty}\sqrt{\frac{1}{n!}\frac{(\ell+n-1)!}{\ell^n(\ell-1)!}}z^n e^{-iF(n)\varphi}|n\rangle \tag{92}$$

with

$$|\mathcal{N}|^2 = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{(\ell+n-1)!}{\ell^n(\ell-1)!}|z|^{2n} = \left(1 - \frac{|z|^2}{\ell}\right)^{-\ell}. \tag{93}$$

The $d\mu$ measure, assumed to be isotropic (i.e., $|z|$-dependent), for which states (92) satisfy the over-completeness relation

$$\int d\mu(|z|)|z,\varphi\rangle\langle z,\varphi| = \sum_{n=0}^{\infty}|n\rangle\langle n| \tag{94}$$

can be easily derived from a reasoning similar to that used for the $su(1,1)$ Perelomov coherent states (see for instance [16, 51]). This leads to

$$d\mu(|z|) = \frac{1}{\pi}\frac{\ell-1}{\ell}\left(1-\frac{|z|^2}{\ell}\right)^{-2} d^2z. \tag{95}$$

There are two other important properties of states (92). First, they are temporally stable, i.e.

$$e^{-iHt}|z,\varphi\rangle = |z,\varphi+t\rangle \tag{96}$$

holds for any real value of t. Second, the $|z,\varphi\rangle$ vector in (92) can be written

$$|z,\varphi\rangle = \mathcal{N}^{-1}\exp(za^+)|0\rangle. \tag{97}$$

Therefore, the $|z,\varphi\rangle$ vectors corresponding to $r=1$ and $\kappa_1 \equiv \kappa = 1/\ell$ with $\ell \in \mathbb{N}^*$ result from the action of a displacement operator on the vacuum and are thus coherent states in the Perelomov sense [42, 51].

5.2 Truncated Fock-Hilbert space

As already mentioned, the norm given by (88) diverges for $r \geq 2$ when $\kappa_i \neq 0$ $(i = 1,2,\cdots,r)$. However, by restricting n to take the values $0,1,\cdots,s-1$ in (88), the norm is defined. This amounts to replace $\mathcal{A}_{\{\kappa\}}$ by the $\mathcal{A}_{\{\kappa\},s}$ truncated algebra and to use the correspondence

$$a^-(s) \longrightarrow \frac{d}{dz} \tag{98}$$

which, for $r \geq 1$ and s finite, can be supplemented with

$$N \longrightarrow z\frac{d}{dz}, \qquad a^+(s) \longrightarrow z\left(1+\kappa_1 z\frac{d}{dz}\right)\left(1+\kappa_2 z\frac{d}{dz}\right)\cdots\left(1+\kappa_r z\frac{d}{dz}\right). \tag{99}$$

Calculations similar to those developed in Section 5.1 lead to the coherent states

$$|z,\varphi\rangle = \mathcal{N}^{-1}\sum_{n=0}^{s-1}\frac{\sqrt{F(n)!}}{n!}z^n e^{-iF(n)\varphi}|n\rangle \tag{100}$$

with

$$|\mathcal{N}|^2 = \sum_{n=0}^{s-1}\frac{F(n)!}{(n!)^2}|z|^{2n} \tag{101}$$

where $F(n)!$ follows from (64) and (17). The states given by (100) and (101) are coherent states in the Perelomov sence since

$$|z,\varphi\rangle = \mathcal{N}^{-1}\exp[za^+(s)]|0\rangle. \tag{102}$$

In addition, they are stable under time evolution.

The situation where

$$\ell_i = \frac{1}{\kappa_i} \in \mathbb{N}^* \quad (i = 1,2,\cdots,r) \tag{103}$$

(each ℓ_i is assumed here to be a strictly positive integer) is especially interesting. In this case, $F(n)$ reads

$$F(n) = \frac{1}{\ell_1\ell_2\cdots\ell_r}n(\ell_1+n-1)(\ell_2+n-1)\cdots(\ell_r+n-1) \tag{104}$$

so that

$$F(n)! = n! \prod_{i=1}^{r} \frac{(\ell_i + n - 1)!}{\ell_i^n (\ell_i - 1)!} = \Gamma(n+1) \prod_{i=1}^{r} \frac{\Gamma(\ell_i + n)}{\ell_i^n \Gamma(\ell_i)}. \tag{105}$$

Consequently, Eq. (101) can be written as

$$|\mathcal{N}|^2 = \sum_{n=0}^{s-1} \frac{1}{n!} \frac{(\ell_1)_n}{\ell_1^n} \frac{(\ell_2)_n}{\ell_2^n} \cdots \frac{(\ell_r)_n}{\ell_r^n} |z|^{2n} \tag{106}$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol. A long calculation, using the inverse Mellin transform [11], shows that the states given by (100) and (106) satisfy the over-completeness relation

$$\int d\mu(|z|) |z,\varphi\rangle\langle z,\varphi| = \sum_{n=0}^{s-1} |n\rangle\langle n| \tag{107}$$

with the measure

$$d\mu(|z|) = \frac{1}{\pi} M(|z|^2) |\mathcal{N}|^2 d^2 z \tag{108}$$

where

$$M(|z|^2) = \frac{\Gamma(\ell_1)\Gamma(\ell_2)\cdots\Gamma(\ell_r)}{\ell_1 \ell_2 \cdots \ell_r} G^{1,0}_{r,1}\left(\frac{|z|^2}{\ell_1 \ell_2 \cdots \ell_r} \middle| \begin{matrix} \ell_1 - 1, \ell_2 - 1, \cdots, \ell_r - 1 \\ 0 \end{matrix} \right) \tag{109}$$

G being the Meijer function defined in [28, 43, 52].

For $r = 1$ and $\kappa_1 = 0$ (corresponding to the limiting case $\ell_1 \to \infty$), Eqs. (100) and (101) give the coherent states

$$|z,\varphi\rangle = \left(\sum_{n=0}^{s-1} \frac{1}{n!} |z|^{2n} \right)^{-1} \sum_{n=0}^{s-1} \frac{1}{\sqrt{n!}} (z e^{-i\varphi})^n |n\rangle \tag{110}$$

which coincide with the coherent states for the Pegg and Barnett oscillator discussed in [34].

5.3 Finite-dimensional Fock-Hilbert space

The search for coherent states

$$|z,\varphi\rangle = \sum_{n=0}^{d-1} \overline{b_n} z^n |n\rangle \tag{111}$$

in the situation where

$$\kappa_1 = -\frac{1}{d-1} < 0 \qquad \kappa_i \geq 0 \quad (i = 2, 3, \cdots, r) \tag{112}$$

can be done on the same pattern as in Sections 5.1 and 5.2 by starting from the correspondence

$$a^- \longrightarrow \frac{d}{dz} \qquad |n\rangle \longrightarrow b_n z^n \equiv \langle \bar{z}, \varphi | n \rangle \tag{113}$$

which is compatible with

$$N \longrightarrow z\frac{d}{dz} \qquad a^+ \longrightarrow z\left(1 - \frac{1}{d-1} z \frac{d}{dz}\right)\left(1 + \kappa_2 z \frac{d}{dz}\right) \cdots \left(1 + \kappa_r z \frac{d}{dz}\right). \tag{114}$$

We thus get

$$|z,\varphi\rangle = \mathcal{N}^{-1} \sum_{n=0}^{d-1} \frac{\sqrt{F(n)!}}{n!} z^n e^{-iF(n)\varphi} |n\rangle \qquad |\mathcal{N}|^2 = \sum_{n=0}^{d-1} \frac{F(n)!}{(n!)^2} |z|^{2n} \tag{115}$$

where $F(n)!$ follows from (64) and (20). The $|z,\varphi\rangle$ states are temporally stable. They satisfy

$$|z,\varphi\rangle = \mathcal{N}^{-1}\exp(za^{+})|0\rangle \tag{116}$$

and are thus coherent states in the Perelomov sense.

In the case

$$\ell_i = \frac{1}{\kappa_i} \in \mathbb{N}^* \quad (i = 2, 3, \cdots, r) \tag{117}$$

the $F(n)!$ generalized factorial in (115) can be calculated to be

$$F(n)! = n!\frac{(d-1)!}{(d-1)^n(d-1-n)!}\prod_{i=2}^{r}\frac{(\ell_i+n-1)!}{\ell_i^n(\ell_i-1)!}. \tag{118}$$

It can be shown, using the inverse Mellin transform [11], that the coherent states given by (115) and (118) satisfy the over-completeness relation

$$\int d\mu(|z|)|z,\varphi\rangle\langle z,\varphi| = \sum_{n=0}^{d-1}|n\rangle\langle n| \tag{119}$$

where the $d\mu$ measure reads

$$d\mu(|z|) = \frac{1}{\pi}M(|z|^2)|\mathcal{N}|^2 d^2z \tag{120}$$

where

$$M(|z|^2) = \frac{1}{(d-1)\Gamma(d)}\frac{\Gamma(\ell_2)\Gamma(\ell_3)\cdots\Gamma(\ell_r)}{\ell_2\ell_3\cdots\ell_r}G_{r,1}^{1,1}\left(\frac{|z|^2}{(d-1)\ell_2\cdots\ell_r}\middle|\begin{matrix}-d,\ell_2-1,\cdots,\ell_r-1\\0\end{matrix}\right) \tag{121}$$

in term of the G Meijer function.

As an example, let us consider the $r=1$ case. From (115) and (118), we have

$$|z,\varphi\rangle = \mathcal{N}^{-1}\sum_{n=0}^{d-1}\sqrt{\frac{1}{n!}\frac{(d-1)!}{(d-1)^n(d-1-n)!}}z^n e^{-iF(n)\varphi}|n\rangle \tag{122}$$

where the normalization factor follows from

$$|\mathcal{N}|^2 = \left(1+\frac{|z|^2}{d-1}\right)^{d-1}. \tag{123}$$

In addition, the $d\mu$ measure is given here by

$$d\mu(|z|) = \frac{1}{\pi}\frac{d}{d-1}\left(1+\frac{|z|^2}{d-1}\right)^{-2} \tag{124}$$

(the expression for $G_{1,1}^{1,1}$ is taken from [43]).

6 *à la* Barut–Girardello coherent states

6.1 Infinite-dimensional Fock-Hilbert space

Going back to $\kappa_i \geq 0$ $(i = 1, 2, \cdots, r)$, we now look for coherent states

$$|z,\varphi\rangle = \sum_{n=0}^{\infty}\overline{c_n}z^n|n\rangle \tag{125}$$

in a realization of $\mathcal{A}_{\{\kappa\}}$ in term of complex variable z with

$$|n\rangle \longrightarrow c_n z^n \equiv \langle \bar{z}, \varphi | n\rangle \qquad a^+ \longrightarrow z \tag{126}$$

(a^+acts as a simple multiplication by z). The combination of (11) with (126) leads to

$$c_n = \sqrt{F(n+1)} e^{-i[F(n+1)-F(n)]\varphi} c_{n+1} \tag{127}$$

which can be iterated to give

$$c_n = c_0 \frac{1}{\sqrt{F(n)!}} e^{+iF(n)\varphi}. \tag{128}$$

We are thus left with states

$$|z, \varphi\rangle = \mathcal{N}^{-1} \sum_{n=0}^{\infty} \frac{1}{\sqrt{F(n)!}} z^n e^{-iF(n)\varphi} |n\rangle \tag{129}$$

with a normalization factor given by

$$|\mathcal{N}|^2 = \sum_{n=0}^{\infty} \frac{1}{F(n)!} |z|^{2n} \tag{130}$$

where $F(n)!$ follows from (64) and (17). The series in (129) and (130) converges in the whole complex plane $\mathbb{C}$. Note that the correspondence (126) can be completed by

$$N \longrightarrow z\frac{d}{dz} \qquad a^- \longrightarrow \left(1 + \kappa_1 z \frac{d}{dz}\right)\left(1 + \kappa_2 z \frac{d}{dz}\right) \cdots \left(1 + \kappa_r z \frac{d}{dz}\right) \frac{d}{dz}. \tag{131}$$

cf. (99) and (114).

The $|z, \varphi\rangle$ vectors are eigenstates of the a^- annihilation operator

$$a^- |z, \varphi\rangle = z |z, \varphi\rangle \tag{132}$$

and can thus be called coherent states in the Barut–Girardello sense [10]. Furthermore

$$e^{-iHt} |z, \varphi\rangle = |z, \varphi + t\rangle \tag{133}$$

which means that they are stable under time evolution.

Let us continue with the special case where each $\ell_i = 1/\kappa_i$ $(i = 1, 2, \cdots, r)$ is a strictly positive integer like in (103). In this case, Eq. (130) can be written as

$$|\mathcal{N}|^2 = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\ell_1^n}{(\ell_1)_n} \frac{\ell_2^n}{(\ell_2)_n} \cdots \frac{\ell_r^n}{(\ell_r)_n} |z|^{2n} \tag{134}$$

or alternatively

$$|\mathcal{N}|^2 = {}_0F_r(\ell_1, \ell_2, \cdots, \ell_r; \ell_1 \ell_2 \cdots \ell_r |z|^2) \tag{135}$$

in term of the generalized hypergeometric function. We can show that the $|z, \varphi\rangle$ coherent states given by Eqs. (129) and (134) satisfy the over-completeness relation

$$\int d\mu(|z|) |z, \varphi\rangle\langle z, \varphi| = \sum_{n=0}^{\infty} |n\rangle\langle n| \tag{136}$$

with the measure

$$d\mu(|z|) = \frac{1}{\pi} M(|z|^2) |\mathcal{N}|^2 d^2 z \tag{137}$$

where

$$M(|z|^2) = \frac{\ell_1 \ell_2 \cdots \ell_r}{\Gamma(\ell_1)\Gamma(\ell_2)\cdots\Gamma(\ell_r)} G^{r+1,0}_{0,r+1}\left(\ell_1\ell_2\cdots\ell_r|z|^2 \middle| 0, \ell_1 - 1, \ell_2 - 1, \cdots, \ell_r - 1\right) \tag{138}$$

in term of the G Meijer function.

Finally, it is interesting to examine the $r = 1$ particular case corresponding to algebra $\mathcal{A}_\kappa$ with $\kappa = 1/\ell$, $\ell \in \mathbb{N}^*$. Then, the coherent states are

$$|z, \varphi\rangle = \mathcal{N}^{-1} \sum_{n=0}^{\infty} \sqrt{\frac{1}{n!} \frac{\ell^n (\ell-1)!}{(\ell+n-1)!}} z^n e^{-iF(n)\varphi} |n\rangle \tag{139}$$

where

$$|\mathcal{N}|^2 = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\ell^n(\ell-1)!}{(\ell+n-1)!} |z|^{2n} = {}_0F_1(\ell; \ell|z|^2). \tag{140}$$

Taking the Meijer function from [43], we have

$$d\mu(|z|) = \frac{1}{\pi} 2\ell K_{\ell-1}(2\sqrt{\ell}|z|) I_{\ell-1}(2\sqrt{\ell}|z|) d^2 z \tag{141}$$

where I and K are modified Bessel functions of first and second kind, respectively.

At this point, a contact with some previous works is in order. First, the states given by (129) and (134) can be viewed as Gazeau–Klauder [35, 41] coherent states associated with a quantum mechanical system whose Hamiltonian is $F(N)$. Second, Eqs. (139)-(141) corresponds to Gazeau-Klauder coherent states for quadratic (in N) Hamiltonians like those for the infinite well and Pöschl-Teller systems [4, 35].

6.2 Finite-dimensional Fock-Hilbert space

6.2.1 Preliminary observation

It is natural to ask if the realization of $\mathcal{A}_{\{\kappa\}}$ of Section 6.1, in which creation operator a^+ acts as multiplication, works in the finite-dimensional case where $n = 0, 1, \cdots, d-1$. The recursion relation (127) is valid for $n = 0, 1, \cdots, d-2$. For $n = d-1$, the action of a^+ on the extremal state $|d-1\rangle$ is zero. Thus, according to (126), coefficient c_{d-1} satisfies

$$c_{d-1} z^d = 0. \tag{142}$$

Equation (142) admits the solution $c_{d-1} = 0$ and from (127) it follows that $c_{d-1} = c_{d-2} = \cdots = c_0 = 0$ leading to trivial and thus unacceptable coherent states. Another solution of (142) is

$$z^d = 0 \tag{143}$$

which is reminiscent of the k-fermions (with $k = d$) discussed in Section 3 and which shows that z cannot be here a complex variable but should be considered as a generalized Grassmann variable of order d. As a conclusion, the construction of coherent states *à la* Barut–Girardello for algebra $\mathcal{A}_{\{\kappa\}}$ with a finite-dimensional representation requires the introduction of Grassmann variables. This point is be developed in the remaining part of this section for the $\mathcal{A}_{\{\kappa\}}$ algebra with $\kappa_1 < 0$ and the $\mathcal{A}_{\{\kappa\},s}$ truncated algebra with $\kappa_1 \geq 0$.

6.2.2 d-fermionic coherent state for $\mathcal{A}_{\{\kappa\}}$

This section concerns the situation where $\kappa_1 < 0$ and $\kappa_i \geq 0$ $(i = 2, 3, \cdots, r)$ for which $F(n)$ is given by (20). Guided by (143) with the z complex variable replaced by a θ generalized Grassmann variable and applying the same *à la* Fock–Bargmann approach as in Section 6.1, we obtain the following unnormalized states

$$|\theta, \varphi\rangle = \sum_{n=0}^{d-1} \frac{1}{\sqrt{F(n)!}} \theta^n e^{-iF(n)\varphi} |n\rangle \tag{144}$$

where $F(n)!$ follows from (64) and (20). The normalization follows from

$$\langle \theta, \varphi | \theta, \varphi \rangle = \sum_{n=0}^{d-1} \exp\left[- i\pi \frac{n(n-1)}{2d} \right] \frac{1}{F(n)!} (\bar{\theta}\theta)^n \tag{145}$$

where Eq. (58) is taken into account.

In view of $\theta^d = 0$, the $|\theta, \varphi\rangle$ states can be called d-fermionic coherent states [18]. They satisfy

$$a^- |\theta, \varphi\rangle = \theta |\theta, \varphi\rangle \tag{146}$$

and are thus coherent states in the Barut–Girardello sense. In addition, they are stable under time evolution, i.e.

$$e^{-iHt} |\theta, \varphi\rangle = |\theta, \varphi + t\rangle. \tag{147}$$

Finally, states (144) constitute an over-complete set with

$$\int |\theta, \varphi\rangle d\mu(\theta, \bar{\theta}) \langle \theta, \varphi| = \sum_{n=0}^{d-1} |n\rangle\langle n| \tag{148}$$

for the $d\mu$ measure satisfying the following integral formula

$$\frac{1}{F(n)!} \int \theta^n \; d\mu(\theta, \bar{\theta}) \; \bar{\theta}^m = \delta_{n,m}. \tag{149}$$

It can be proved that $d\mu$ is given by

$$d\mu(\theta, \bar{\theta}) = \sum_{n=0}^{d-1} F(n)! \; d\theta \; \theta^{d-1-n} \; \bar{\theta}^{d-1-n} \; d\bar{\theta} \tag{150}$$

owing to integration rules (55) and (56).

As an example, let us take $r = 1$ corresponding to $\mathcal{A}_\kappa$ with $\kappa \equiv \kappa_1 < 0$ and $-1/\kappa \in \mathbb{N}^*$. The latter algebra can be identified to $su(2)$ with generators J_-, J_+ and J_3 [8]

$$J_- = \frac{1}{\sqrt{-\kappa}} a^- \qquad J_+ = \frac{1}{\sqrt{-\kappa}} a^+ \qquad J_3 = \frac{1}{2\kappa}(I + 2\kappa N). \tag{151}$$

By making the substitutions

$$|n\rangle \longleftrightarrow |j, m\rangle \qquad n \longleftrightarrow j + m \qquad -\frac{1}{\kappa} \longleftrightarrow 2j \qquad d \longleftrightarrow 2j + 1 \tag{152}$$

where $m = -j, -j+1, \cdots, +j$ in terms of angular momentum or $su(2)$ quantum numbers, Eq. (144) can be specialized to

$$|\theta, \varphi\rangle = \sum_{m=-j}^{+j} \sqrt{\frac{1}{(2j)!} \frac{(j-m)!}{(j+m)!}} \left(\sqrt{2j}\theta\right)^{j+m} e^{-iF(j+m)\varphi} |j, m\rangle. \tag{153}$$

These states satisfy the eigenvalue equation

$$J_- |\theta, \varphi\rangle = \sqrt{2j}\theta |\theta, \varphi\rangle \tag{154}$$

and thus can be viewed as $su(2)$ Barut–Girardello coherent states labeled by a θ generalized Grassmann variable of order $2j+1$ ($\theta^{2j+1} = 0$).

Two extremal cases are of interest. First for $d = 2$, Eq. (153) gives the states

$$|\theta, \varphi\rangle = |1/2, -1/2\rangle + \theta e^{-i\varphi} |1/2, +1/2\rangle = |0\rangle + \theta e^{-i\varphi} |1\rangle \tag{155}$$

which for $\varphi = 0$ coincide with the coherent states for the fermionic oscillator [13] or a qubit. Second in the limiting case $d \to \infty$ (or $\kappa \to 0^-$), we can identify θ to a complex variable, say z, and (144) leads to the states

$$|z, \varphi\rangle = \sum_{n=0}^{\infty} \sqrt{\frac{1}{n!}} z^n e^{-in\varphi} |n\rangle \tag{156}$$

which look like the Glauber coherent states for the bosonic oscillator up to a re-labeling of $z \exp(-i\varphi)$ as z.

6.2.3 s-fermionic coherent state for $\mathcal{A}_{\{\kappa\},s}$

For the $\mathcal{A}_{\{\kappa\},s}$ truncated algebra with $\kappa_i \geq 0$ ($i = 1, 2, \cdots, r$), the approach developed in 6.2.2 for deriving Barut–Girardello coherent states is valid under the condition to replace d by s, a^- by $a^-(s)$ and $F(n)$ calculated via (20) by $F(n)$ calculated via (17). This leads to coherent states of type (144) with $d = s$.

As an example, let us consider the case where $r = 1$ and $\kappa \equiv \kappa_1 > 0$. The corresponding algebra, $\mathcal{A}_\kappa$ can be identified with $su(1,1)$ with generators K_-, K_+ and K_3 given by [8]

$$K_- = \frac{1}{\sqrt{\kappa}} a^- \qquad K_+ = \frac{1}{\sqrt{\kappa}} a^+ \qquad K_3 = \frac{1}{2\kappa}(I + 2\kappa N). \tag{157}$$

To pass from the representation of $\mathcal{A}_\kappa$ to that of $su(1,1)$ we make the substitutions

$$|n\rangle \longleftrightarrow |b, b+n\rangle \qquad n \longleftrightarrow n \qquad \frac{1}{\kappa} \longleftrightarrow 2b \tag{158}$$

where b is the Bargmann index (generally noted k but here noted b in order to avoid confusion with k in k-fermions). The truncation to order s of the infinite-dimensional discrete representation of $su(1,1)$ yields the coherent states

$$|\theta, \varphi\rangle = \sum_{n=0}^{s-1} \sqrt{\frac{(2b-1)!}{n!(2b+n-1)!}} \left(\sqrt{2b}\theta\right)^{n} e^{-iF(n)\varphi} |b, b+n\rangle \tag{159}$$

which satisfy

$$K_-(s)|\theta,\varphi\rangle = \sqrt{2b}\theta|\theta,\varphi\rangle \tag{160}$$

where $K_-(s) = \sqrt{2b}a^-(s)$.

We close this section with the truncated oscillator algebra. This algebra is described by Eqs. (25)-(31) with $F(n) = n$ ($\kappa_i = 0$ for $i = 1, 2, \cdots, r$). The corresponding s-fermionic coherent states are simply

$$|\theta,\varphi\rangle = \sum_{n=0}^{s-1} \frac{1}{\sqrt{n!}} \theta^n e^{-in\varphi} |n\rangle. \tag{161}$$

They satisfy

$$a^-(s)|\theta,\varphi\rangle = \theta|\theta,\varphi\rangle \tag{162}$$

and can be refered to Barut–Girardello coherent states for the Pegg–Barnett oscillator.

7 Closing remarks

The $\mathcal{A}_{\{\kappa\}}$ polynomial Weyl-Heisenberg algebra considered in this paper generalizes, along the line of the works in [17, 20, 21, 22, 24, 29, 32, 38, 48, 53], the $\mathcal{A}_\kappa$ algebra introduced in [22]. Indeed, the $\mathcal{A}_{\{\kappa\}}$ algebra, characterized by structure function F such that $F(N)$ is a polynomial of order $r+1$ in number operator N, gives for $r = 1$ the $\mathcal{A}_\kappa$ algebra which gives in turn the $su(2)$, $su(1,1)$ and oscillator algebras as very special cases. Note that the $\mathcal{A}_{\{\kappa\}}$ algebra can be viewed as a special class of the f-oscillators introduced by Man'ko *et al* [48]. The dimension of the Fock-Hilbert representation space of $\mathcal{A}_{\{\kappa\}}$, with $\{\kappa\} \equiv \{\kappa_1, \kappa_2, \cdots, \kappa_r\}$, depends on the sign of the κ_i parameters ($i = 1, 2, \cdots, r$). In the case of an infinite-dimensional space, we used a truncation procedure of the Pegg–Barnett type for generating a $\mathcal{A}_{\{\kappa\},s}$ truncated algebra of truncation order s. As a result, of importance for deriving certain coherent states, a connection was established between the s-fermionic algebra (arising from two quon algebras) and either the $\mathcal{A}_{\{\kappa\},s}$ truncated algebra or the $\mathcal{A}_{\{\kappa\}}$ algebra with a representation of finite dimension ($d = s$). Note that such a connection can be derived between any generalized Weyl-Heisenberg algebra with a representation of dimension $d = s$ and the s-fermionic algebra.

Very few papers were devoted to the construction of coherent states of the Perelomov type (i.e., resulting from the action of a displacement operator on the vacuum) for generalized Weyl-Heisenberg algebras in finite or infinite dimension. We may mention Ref. [26] corresponding to a polynomial Weyl-Heisenberg algebra with $r = 1$. The approach developed in [26] is very difficult to adapt to the case $r \geq 2$. In the present paper, Perelomov coherent states were worked out in a Fock–Bargmann realization where the annihilation operator of $\mathcal{A}_{\{\kappa\}}$ and $\mathcal{A}_{\{\kappa\},s}$ acts as derivative by a bosonic complex variable. For a infinite-dimensional representation of $\mathcal{A}_{\{\kappa\}}$, it was shown that bosonic Peremolov coherent states exist if and only if $r = 1$. For a finite-dimensional representation of $\mathcal{A}_{\{\kappa\}}$ or $\mathcal{A}_{\{\kappa\},s}$, bosonic Perelomov coherent states were constructed for arbitrary r; to the best of our knowledge, this constitutes the first derivation of Peremolov coherent states in finite dimension.

The situation is different for coherent states of the Barut–Girardello type. Most of the works on Barut–Girardello coherent states were achieved for infinite-dimensional representations of generalized

Weyl-Heisenberg algebras by diagonalizing an annihilation operator (for example, see [29, 48]). In this paper, bosonic Barut–Girardello coherent states for an infinite-dimensional representation of $\mathcal{A}_{\{\kappa\}}$ were derived in a Fock–Bargmann realization where the creation operator acts as multiplication by a bosonic complex variable. For finite-dimensional representation of $\mathcal{A}_{\{\kappa\}}$ and $\mathcal{A}_{\{\kappa\},s}$, the construction of Barut–Girardello coherent states required the introduction of generalized Grassmann variables (with the creation operator acting as multiplication by a k-fermionic or generalized Grassmann variable). The construction of Barut–Girardello coherent states in finite dimension (illustrated with the $su(2)$ algebra, the $su(1,1)$ truncated algebra and the Pegg–Barnett oscillator algebra) is realized in this article for the first time.

The (bosonic) Perelomov coherent states as well as the (bosonic and k-fermionic) Barut-Girardello coherent states constructed in this work satisfy continuity, temporal stability under time evolution and an over-completeness property. For bosonic Perelomov and Barut–Girardello coherent states, the over-completeness property was derived by means of a measure using the G Meijer function and for k-fermionic Barut–Girardello coherent states the relevant measure was obtained by using the Majid–Rodríguez-Plaza integration formula for generalized Grassmann variables.

As a possible extension of this work, let us mention the derivation of those states of $\mathcal{A}_{\{\kappa\}}$ which minimize the Robertston-Schrödinger [56, 57] uncertainty relation. It would be interesting to see how these states (called intelligent states in the cases of $su(2)$ and $su(1,1)$ [6, 60, 61]) incorporate in a unified scheme Perelomov and Barut–Girardello states. Moreover, another interesting extension would be to construct coherent state vectors, in the sense of Ali–Engliš–Gazeau [3], for the $\mathcal{A}_{\{\kappa\}}$ and $\mathcal{A}_{\{\kappa\},s}$ algebras.

It should be emphasized that most of the generalized Weyl-Heisenberg algebras considered in the literature have infinite-dimensional representations. The consideration in the present paper of generalized Weyl-Heisenberg algebras with finite-dimensional representations, along the line initiated in [22, 23], opens an avenue of future developments, especially in quantum information and quantum computation where finite-dimensional Hilbert spaces are of paramount importance. We hope that the results contained in this paper (in particular, the Barut–Girardello coherent states in finite dimension) would be useful in quantum information where intrication of coherent states plays a fundamental role. As encouraging preliminary works, the $\mathcal{A}_{\kappa}$ algebra already proved to be useful [22, 23], for deriving mutually unbiased bases of interest in quantum cryptography and quantum state tomography, and intrication of k-fermionic coherent states (defined in [18]) was studied in [47].

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