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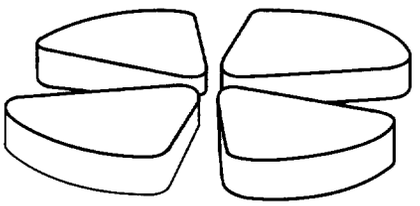

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GANIL P 97 28

SYMMETRIES IN NUCLEAR MODELS

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Some general notions of symmetry and dynamical symmetry in quantum mechanics are introduced. It is then indicated how techniques based on symmetry considerations can be used to find the eigensolutions of an aggregate of interacting particles (bosons and/or fermions) in analytical form. The symmetries of the spherical nuclear shell model are discussed in detail.

1 Introduction

The lectures I gave at the VI Hispalensis International Summer School in Oromana (near Sevilla) on *Symmetries in nuclear models* consisted of three parts:

1. Symmetries and their applications in physics;
2. Symmetries in the spherical shell model;
3. Symmetries in the interacting boson model.

I felt that, due to limitations in length, it was not possible to present a detailed written account of the three lectures. Rather than giving a brief summary of each lecture, I chose to present one in full, savagely abbreviate another and completely neglect the third. Given the topic of the school, *Many-Body Theory of Correlated Fermion Systems*, the natural choice for the written text was symmetries in the nuclear shell model, discussed here in section 3. It is preceded by a section on the general role of symmetries in quantum mechanics, which presents some elements of the first lecture necessary for the comprehension of symmetries in nuclear models. A review article on symmetries in nuclear models which will include all the material discussed in the lectures (and more) is in preparation [1].

Detailed discussions of group theory with applications in physics such as I presented in the first lecture can be found in many monographs. An admirable example is the book by Elliott and Dawber [2]; an account of the use of group theory in molecular and nuclear physics is also given by Frank and myself[3]. The group-theoretical structure of the interacting boson model has been presented at numerous occasions. Suffice it to cite here the monograph on the interacting boson model [4] and the one on the interacting boson–fermion model [5] dealing with systems of bosons, and bosons and fermions, respectively. Even more

text books exist on the nuclear shell model. The ones that I find myself using frequently are by Brussaard and Glaudemans[6], Lawson[7], Heyde[8] and Talmi[9]. With the exception of Talmi's beautiful and comprehensive monograph, not so much attention is paid usually to symmetries in the shell model. I tried in the second lecture, as well as in the present text, to expound this topic from a unified perspective such that the different symmetries (Racah's pairing model, Elliott's rotation model, etc.) arise as particular limits of single shell-model hamiltonian. This, I believe, is an original approach which is appealing from a pedagogical point of view. It is also the feature that convinced me to place emphasis on the shell-model part of the lectures.

2 Symmetry in Quantum Mechanics

First some elementary considerations related to *symmetry* will be given, mainly to introduce the notation adopted in these notes. For applications in low-energy nuclear physics, the idea of symmetry is most conveniently introduced via a hamiltonian formalism. A hamiltonian $\hat{H}$, invariant under a set of transformations $\hat{g}_i$ which together form a Lie group G, i.e.

$$[\hat{H}, \hat{g}_i] = 0, \qquad \text{for } \hat{g}_i \in G, \tag{1}$$

is said to have a symmetry G or, alternatively, to be *invariant under* G. A well-known consequence of a symmetry is the occurrence of degeneracies in the eigenspectrum of $\hat{H}$. Given an eigenstate ψ of $\hat{H}$ with energy E, the condition (1) implies that the states $\hat{g}_i\psi$ have the same energy. An arbitrary eigenstate of $\hat{H}$ can thus be written as $|\Gamma\gamma\rangle$, where the first quantum number Γ is different for states with different energies and the second quantum number γ is needed to label degenerate eigenstates. The eigenvalues of a hamiltonian that satisfies (1) depend on Γ only,

$$\hat{H}|\Gamma\gamma\rangle = E(\Gamma)|\Gamma\gamma\rangle, \tag{2}$$

and, furthermore, the transformations $\hat{g}_i$ do not admix states with different Γ:

$$\hat{g}_i|\Gamma\gamma\rangle = \sum_{\gamma'} a^{\Gamma}_{\gamma'\gamma}(i)|\Gamma\gamma'\rangle. \tag{3}$$

In the language of group theory the transformation coefficients $a^{\Gamma}_{\gamma'\gamma}(i)$, when considered as matrices in the indices γ and γ', provide a (matrix) *representation* of the group elements $\hat{g}_i$ in the vector space spanned by the states $|\Gamma\gamma\rangle$. A representation obviously depends on Γ and is denoted as $[\Gamma]$. Another ingredient borrowed from group theory concerns the construction of operators like $\hat{H}$ in (1) that commute with all elements of G. Such operators are called

Casimir operators and are denoted here as $\hat{C}_n[G]$, the index n referring to the order of the operator in the $\hat{g}_i$.

The concept of a *dynamical symmetry* can now be introduced, for which (at least) two groups G_1 and G_2 with $G_1 \supset G_2$ are needed. Imposing the condition of G_1 symmetry on the hamiltonian $\hat{H}$, its eigenstates can, as before, be labelled as $|\Gamma_1\gamma_1\rangle$. But, since $G_1 \supset G_2$, a hamiltonian with G_1 symmetry necessarily must also have a symmetry G_2 and, consequently, its eigenstates can also be labelled as $|\Gamma_2\gamma_2\rangle$. Combination of the two properties leads to the eigenequation [a]

$$\hat{H}|\Gamma_1\Gamma_2\gamma_2\rangle = E(\Gamma_1)|\Gamma_1\Gamma_2\gamma_2\rangle, \tag{4}$$

where the rôle of γ_1 is played by $\Gamma_2\gamma_2$. Hence the eigenvalues depend only on Γ_1. The meaning of the labels used in (4) is further illustrated with the transformation properties of the states $|\Gamma_1\Gamma_2\gamma_2\rangle$ under the action of an element belonging to G_1 or G_2:

$$\begin{aligned} \hat{g}_i|\Gamma_1\Gamma_2\gamma_2\rangle &= \sum_{\Gamma_2'\gamma_2'} a^{\Gamma_1}_{\Gamma_2'\gamma_2'\Gamma_2\gamma_2}(i)|\Gamma_1\Gamma_2'\gamma_2'\rangle, \qquad \text{for } \hat{g}_i \in G_1, \\ \hat{g}_i|\Gamma_1\Gamma_2\gamma_2\rangle &= \sum_{\gamma_2'} a^{\Gamma_2}_{\gamma_2'\gamma_2}(i)|\Gamma_1\Gamma_2\gamma_2'\rangle, \qquad \text{for } \hat{g}_i \in G_2. \end{aligned} \tag{5}$$

In many applications the condition of G_1 symmetry is found to be too strong. A *possible* breaking of the G_1 symmetry occurs via the hamiltonian

$$\hat{H}' = a\hat{C}_{n_1}[G_1] + b\hat{C}_{n_2}[G_2]. \tag{6}$$

The idea is to take a combination of Casimir operators of G_1 *and* G_2. The symmetry properties of the hamiltonian $\hat{H}'$ are now as follows. Since $[\hat{H}', \hat{g}_i] = 0$ for $\hat{g}_i \in G_2$, $\hat{H}'$ is invariant under G_2. The hamiltonian $\hat{H}'$ generally does not, however, commute with all elements of G_1 and for this reason the G_1 symmetry is broken, the extent of the symmetry breaking depending on the ratio b/a. Furthermore, since $\hat{H}'$ is a combination of Casimir operators of G_1 and G_2, its eigenvalues are obtained in closed form:

$$\Big(a\hat{C}_{n_1}[G_1] + b\hat{C}_{n_2}[G_2]\Big)|\Gamma_1\Gamma_2\gamma_2\rangle = \Big(aE_{n_1}(\Gamma_1) + bE_{n_2}(\Gamma_2)\Big)|\Gamma_1\Gamma_2\gamma_2\rangle. \tag{7}$$

The conclusion is thus that, although $\hat{H}'$ is not invariant under G_1, its eigenstates are the same as those of $\hat{H}$ in (4). The hamiltonian $\hat{H}'$ is said to

[a] In (4) the representation $[\Gamma_2]$ is assumed to occur only once in $[\Gamma_1]$, otherwise an additional quantum number α would have been needed to uniquely label the states as $|\Gamma_1\alpha\Gamma_2\gamma_2\rangle$. For the purpose of illustrating the concept of dynamical symmetry, however, this technical complication can be ignored.

have G_1 as a dynamical symmetry. The essential feature is that, although the eigenvalues of $\hat{H}'$ depend on Γ_1 *and* Γ_2 (and hence G_1 is not a symmetry), the eigenstates do not change during the breaking of the G_1 symmetry: the dynamical symmetry breaking splits but does not admix the eigenstates.

The most important consequence of a symmetry, which remains valid under the process of a dynamical symmetry breaking, is the existence of quantum numbers. Frequently these quantum numbers give rise to selection rules in electromagnetic transition or particle transfer processes; the measurement of transition or transfer probabilities is thus a method to establish the goodness of quantum numbers and this in turn indicates to what extent a given (dynamical) symmetry is realised.

The link between symmetries and selection rules, expressed above in a qualitative manner, can be given a precise quantitative formulation through a generalisation of the Wigner–Eckart theorem. This theorem is well known for the case SU(2) $\supset$ O(2) with associated labels of angular momentum J and its projection M_J. The generalisation involves an arbitrary labelling of the type

$$\begin{array}{ccc} G_1 & \supset & G_2 \\ \downarrow & & \downarrow \\ \Gamma & & \gamma \end{array}, \tag{8}$$

where below each algebra the associated label is given. Suppose the calculation is required of a transition or transfer matrix element between an initial state $|\Gamma_i\gamma_i\rangle$ and a final state $|\Gamma_f\gamma_f\rangle$. To find the corresponding matrix element it is first necessary to determine the tensor character of the operator associated with the transition or transfer which generally is achieved by writing the operator as $\sum_{\Gamma\gamma}\hat{T}(\Gamma\gamma)$. Each piece $\hat{T}(\Gamma\gamma)$ can now be dealt with separately by the *generalised Wigner–Eckart theorem* which states that[b]

$$\langle\Gamma_f\gamma_f|\hat{T}(\Gamma\gamma)|\Gamma_i\gamma_i\rangle = \langle\Gamma_i\gamma_i\ \Gamma\gamma|\Gamma_f\gamma_f\rangle\langle\Gamma_f\parallel\hat{T}(\Gamma)\parallel\Gamma_i\rangle. \tag{9}$$

From this result it is seen that the matrix element can be written as the product of a *generalised coupling coefficient* (denoted as $\langle\cdots\cdots|\cdots\rangle$) and a reduced matrix element (written as $\langle\cdot\parallel\cdot\parallel\cdot\rangle$). The essential point is that all dependence on the quantum numbers associated with the subalgebra G_2 is entirely contained in the generalised coupling coefficient. The calculation of the latter is a purely algebraic problem that boils down to a matrix diagonalisation. In addition, selection rules follow from (9): if $[\Gamma_f]$ is not contained in the product $[\Gamma_i]\times[\Gamma]$ the generalised coupling coefficient is zero and the matrix element vanishes.

[b]Again technical complications due to multiplicities are ignored here for simplicity. The technical complications in this case can be twofold: i) $[\Gamma_f]$ may occur more than once in the product $[\Gamma_i]\times[\Gamma]$; ii) $[\gamma]$ may be contained more than once in $[\Gamma]$.

3 Symmetries of the Spherical Nuclear Shell Model

3.1 A Simple Nuclear Shell Model

The nuclear structure problem consists in the solution of the A-body Schrödinger equation

$$\hat{H}\Psi(1,2,\ldots,A) = E\Psi(1,2,\ldots,A), \tag{10}$$

where the hamiltonian $\hat{H}$,

$$\hat{H} = \sum_{k=1}^{A} \hat{T}(k) + \sum_{k<l=1}^{A} \hat{W}(k,l), \tag{11}$$

consists of a kinetic energy $\hat{T}(k)$ of nucleon k and a two-body interaction $\hat{W}(k,l)$ between nucleons k and l. By adding a *mean-field* potential $\hat{V}(k)$ to the kinetic energy and substracting the same from the interaction, this hamiltonian is rewritten in two parts, the first of which defines the *independent-particle* motion while the second is the *residual interaction*,

$$\begin{aligned} \hat{H} &= \sum_{k=1}^{A} \Big(\hat{T}(k) + \hat{V}(k)\Big) + \left(\sum_{k<l=1}^{A} \hat{W}(k,l) - \sum_{k=1}^{A} \hat{V}(k)\right) \\ &\equiv \hat{H}_{\rm IP} + \sum_{k<l=1}^{A} \hat{W}'(k,l). \end{aligned} \tag{12}$$

A judicious choice of the mean-field potential $\hat{V}(k)$ will minimise the effect of the residual interaction and will have the consequence that the true nuclear wavefunction can be well represented by a Slater determinant built from the single-particle solutions of the potential $\hat{V}$. The effect of $\hat{W}'(k,l)$ can never be completely neglected even for the optimum choice of mean-field potential and generally additional correlation effects are required through the appropriate associated residual interaction.

A simple but reasonably adequate form of the mean-field potential consists of a three-dimensional harmonic oscillator corrected with a spin–orbit and an orbit–orbit term,

$$\hat{V}(k) = \tfrac{1}{2} m\omega^2 r_k^2 + v_{ls}\vec{l}_k \cdot \vec{s}_k + v_{ll}\vec{l}_k \cdot \vec{l}_k, \tag{13}$$

where m is the nucleon mass and ω the angular frequency, assumed to be identical for neutrons and protons. An attractive spin–orbit term ($v_{ls} < 0$) is required to achieve shell closures at the magic numbers. An orbit–orbit term

with $v_{ll} < 0$ pushes down single-particle states with high orbital angular momentum l which is one of the main effects if the schematic harmonic-oscillator mean-field is substituted for the more realistic Woods–Saxon potential.

To summarise, the simplified spherical shell-model hamiltonian that is used in this section is of the form

$$\hat{H}_{\rm SM} = \sum_{k=1}^{A} \left(\frac{p_k^2}{2m} + \tfrac{1}{2} m\omega^2 r_k^2 + v_{ls} \vec{l}_k \cdot \vec{s}_k + v_{ll} \vec{l}_k \cdot \vec{l}_k \right) + \sum_{k<l=1}^{A} \hat{W}'(k,l). \quad (14)$$

The question that arises in connection with this hamiltonian is whether, for some special choice of mean field and interaction, it exhibits (dynamical) symmetries that lead to an analytical solution. Over the years, the existence of three such limits has been recognised:

- *No residual interaction.* If $\hat{W}' = 0$ the solution of (14) reduces to a Slater determinant built from harmonic-oscillator eigenstates.

- *Pairing interaction in jj coupling.* In case of a strong spin–orbit interaction a jj-coupling scheme is obtained. If, in addition, the residual interaction has a pairing character Racah's SU(2) seniority classification follows.

- *Quadrupole interaction in LS coupling.* In case of a weak spin–orbit interaction an LS-coupling or Russel–Saunders coupling is obtained. If, in addition, the residual interaction has a quadrupole character Elliott's SU(3) model of rotation follows.

The first case yields an uncorrelated many-fermion wavefunction and, given the topic of this school, is not discussed further here. The properties of the second and third cases are analysed in the subsection 3.2 and 3.4. Elliott's classification concerns the orbital part of the many-fermion wavefunction but presupposes Wigner's SU(4) symmetry of the isospin–spin part of the wavefunction. The latter is explained in subsection 3.3 as a prelude to Elliott's model.

3.2 Racah's Quasi-Spin SU(2) Model of Pairing

If the coefficient v_{ls} of the spin–orbit term in the mean-field potential (13) is sufficiently large a jj-coupling scheme arises. Suppose initially that the splitting between the different j states is big compared to the residual interaction and suppose, furthermore, that the residual interaction has a pairing character,

that is, is of the form

$$\langle j^2 J M_J | \hat{W}'_{\text{pairing}} | j^2 J M_J \rangle = \begin{cases} -\frac{1}{2}(2j+1)G, & \text{if } J = 0 \\ 0, & \text{if } J \neq 0 \end{cases} . \tag{15}$$

This is a reasonable albeit schematic approximation to the residual interaction between two *identical* nucleons (i.e., for $T = 1$ but not for $T = 0$) and hence can only be valid in semi-magic nuclei. Under the above assumptions the schematic shell-model hamiltonian (14) reduces to

$$\hat{H}_{\text{SM}} = E_0 + \hat{W}'_{\text{pairing}}, \tag{16}$$

where E_0 represents a constant energy contribution from all nucleons in shells below the valence j shell.

The hamiltonian (16) can be diagonalised analytically [10] in a space of n identical fermions in a j shell by noting the second-quantised form of the pairing interaction,

$$\hat{W}'_{\text{pairing}} = -G\hat{S}_+\hat{S}_-, \tag{17}$$

with

$$\hat{S}_+ = \tfrac{1}{2}\sqrt{2j+1}[a_j^\dagger \times a_j^\dagger]_0^{(0)}, \qquad \hat{S}_- = \left(\hat{S}_+\right)^\dagger, \tag{18}$$

where $a_j^\dagger$ creates a particle in the j shell. If, in addition to $\hat{S}_\pm$, also the operator

$$\hat{S}_z = \tfrac{1}{4}(2\hat{n}_j - 2j - 1) = \tfrac{1}{4}\left(\sum_{m=-j}^{j} a_{jm}^\dagger a_{jm} - 2j - 1\right) \tag{19}$$

is considered, then the following commutation relations hold:

$$[\hat{S}_z, \hat{S}_\pm] = \pm\hat{S}_\pm, \qquad [\hat{S}_+, \hat{S}_-] = 2\hat{S}_z. \tag{20}$$

This shows that the set of operators $\{\hat{S}_z, \hat{S}_+, \hat{S}_-\}$ forms an SU(2) algebra, which is referred to as the *quasi-spin* algebra. By virtue of this relation with the quasi-spin SU(2) algebra the pairing hamiltonian can now be readily solved. Since

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \tfrac{1}{2}(\hat{S}_+\hat{S}_- + \hat{S}_-\hat{S}_+) + \hat{S}_z^2 = \hat{S}_+\hat{S}_- + \hat{S}_z^2 - \hat{S}_z, \tag{21}$$

the pairing interaction can be rewritten as

$$\hat{S}_+\hat{S}_- = \hat{S}^2 - \hat{S}_z^2 + \hat{S}_z. \tag{22}$$

This shows that the pairing hamiltonian (17) has an SU(2) $\supset$ O(2) dynamical symmetry with the eigensolutions $|sm_s\rangle$ and eigenvalues

$$\hat{S}_+\hat{S}_-|sm_s\rangle = [s(s+1) - m_s(m_s-1)]|sm_s\rangle. \tag{23}$$

Converting to the customary labels of particle number n and *seniority* [11] v through

$$s = \tfrac{1}{4}(2j - 2v + 1), \qquad m_s = \tfrac{1}{4}(2n - 2j - 1), \tag{24}$$

the eigensolutions of the hamiltonian (16) can be given as

$$(E_0 - G\hat{S}_+\hat{S}_-)|j^n vJM_J\rangle = \left(E_0 - \tfrac{1}{4}G(n-v)(2j-n-v+3)\right)|j^n vJM_J\rangle, \tag{25}$$

where also the total angular momentum J and its projection M_J can be given since the hamiltonian (16) is rotationally invariant. The interpretation of the eigensolutions (25) can now be found out by simple action of the $\hat{S}_\pm$ operators. For example, from (25) it follows that the action of $\hat{S}_-$ on a state with $n = v$ gives zero:

$$\hat{S}_-|j^v vJM_J\rangle = 0. \tag{26}$$

The interpretation of this result is that the state $|j^v vJM_J\rangle$ contains no pairs of particles coupled to zero. Furthermore, it can be shown that the action of the operator $\hat{S}_+$ does not change the seniority quantum number while it obviously increases the particle number by two:

$$|j^n vJM_J\rangle \propto \left(\hat{S}_+\right)^{(n-v)/2}|j^v vJM_J\rangle. \tag{27}$$

Hence the interpretation of the seniority quantum number v is that it gives the number of nucleons not in pairs coupled to angular momentum zero.

The above results are valid for identical nucleons in a single j orbit interacting through a pairing force. In fact, it can be easily generalised to a situation of identical nucleons in many degenerate orbits by making the substitution

$$\hat{S}_+ \rightarrow \sum_j \hat{S}_+(j), \tag{28}$$

leaving unchanged all previous results, such as the algebraic structure (20). The ensuing quasi-spin formalism can then be applied to semi-magic nuclei and requires the assumption that the pairing interaction is strong as compared to the splitting between the j orbits in the valence shell but that it is too weak to break up the core.

3.3 Wigner's Isospin-Spin SU(4) Symmetry

Some five years after Heisenberg's suggestion [12] of isospin symmetry in nuclei, Wigner [13] (and, independently of him, Hund [14]) proposed a beautiful extension of this idea by assuming nuclear forces to be invariant under rotations in *spin* as well as *isospin* space. This invariance is expressed by the following commutation relations:

$$[\hat{H}, \sum_{k=1}^{A} \hat{t}_\mu(k)] = [\hat{H}, \sum_{k=1}^{A} \hat{s}_\mu(k)] = [\hat{H}, \sum_{k=1}^{A} \hat{t}_\mu(k)\hat{s}_\nu(k)] = 0, \tag{29}$$

where $\hat{t}_\mu(k)$ and $\hat{s}_\mu(k)$ are the isospin and spin components, respectively, of particle k and $\mu, \nu = 0, \pm 1$ are their projections on the z axis. The summation extends over all particles in the nucleus. The 15 operators $\sum_k \hat{t}_\mu(k)$, $\sum_k \hat{s}_\mu(k)$ and $\sum_k \hat{t}_\mu(k)\hat{s}_\nu(k)$ together form an SU(4) Lie algebra, denoted here as $\mathrm{SU}_{TS}(4)$, and thus, according to the discussion in section 2, any hamiltonian satisfying the conditions (29) has SU(4) symmetry. Because of the first two conditions in (29), the eigenstates of any such hamiltonian must also have total isospin T and total spin S as good quantum numbers. Furthermore, the additional requirement of rotational invariance,

$$[\hat{H}, \sum_{k=1}^{A} \hat{j}_\mu(k)] \equiv [\hat{H}, \sum_{k=1}^{A} (\hat{l}_\mu(k) + \hat{s}_\mu(k))] = 0, \tag{30}$$

then also implies the goodness of total orbital angular momentum L since the combination of (29) and (30) yields

$$[\hat{H}, \sum_{k=1}^{A} \hat{l}_\mu(k)] = 0. \tag{31}$$

As a consequence of the invariances (29) the A-particle eigenstates of $\hat{H}$ can be written as

$$\Psi(1, 2, \ldots, A) = \Psi_{(\lambda\mu\nu)LM_L}(1, 2, \ldots, A)\Psi_{TM_TSM_S}(1, 2, \ldots, A), \tag{32}$$

where the coupling of L and S to total J and M_J is not shown. The first part depends on the spatial coordinates $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_A$ of the A nucleons while the second part involves their isospin–spin coordinates. Besides the labels L, T and S associated with [c] $\mathrm{O}_L(3)$, $\mathrm{SU}_T(2)$ and $\mathrm{SU}_S(2)$, and the magnetic sublabels

[c] Since L is always integer while T and S can be both integer and half-integer, the associated rotational invariances are denoted by O(3) in the former and by SU(2) in the latter case.

Table 1: Classification of one and two particles in the *sd* shell

n	L	(T,S)	$[f_1'f_2'f_3'f_4']$	$[f_1f_2f_3f_4]$
1	$0,2$	$(\frac{1}{2},\frac{1}{2})$	$\square = [1000]$	$\square = [1000]$
2	$0^2,2^2,4$	$(0,1),(1,0)$	$\square\square = [2000]$	$\begin{smallmatrix}\square\\\square\end{smallmatrix} = [1100]$
	$1,2,3$	$(0,0),(1,1)$	$\begin{smallmatrix}\square\\\square\end{smallmatrix} = [1100]$	$\square\square = [2000]$

associated with their respective O(2) subalgebras, there appear in (32) also the labels $(\lambda\mu\nu)$ connected with $\mathrm{SU}_{TS}(4)$. They can be defined mathematically in terms of $\mathrm{U}_{TS}(4)$ Young tableaux with lengths f_1, f_2, f_3 and f_4 which are related to $(\lambda\mu\nu)$ through $\lambda = f_1 - f_2$, $\mu = f_2 - f_3$ and $\nu = f_3 - f_4$. The labelling of eigenstates of $\hat{H}$ can be summarised conveniently as

$$\begin{array}{ccccccccc} \mathrm{U}(4\omega) & \supset & \Big(\mathrm{U}_L(\omega) & \supset & \cdots & \supset & \mathrm{O}_L(3)\Big) & & \\ \downarrow & & \downarrow & & & & \downarrow & & \\ [1^n] & & [f_1'f_2'f_3'f_4'] & & & & L & & \end{array}$$

$$\otimes \quad \begin{array}{ccccccccc} \Big(\mathrm{U}_{TS}(4) & \supset & \mathrm{SU}_{TS}(4) & \supset & \mathrm{SU}_T(2) & \otimes & \mathrm{SU}_S(2)\Big) & & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ [f_1f_2f_3f_4] & & (\lambda\mu\nu) & & T & & S & & \end{array} , \tag{33}$$

which is appropriate for n valence nucleons in a harmonic-oscillator shell with degeneracy ω (i.e., $\omega = 1,3,6,\ldots$ for the $s, p, sd, \ldots$ shells). The complete wavefunction Ψ is totally antisymmetric, which is expressed by total antisymmetry $[1^n]$ under $\mathrm{U}(4\omega)$. The separate wavefunctions Ψ_{LM_L} and $\Psi_{TM_TSM_S}$, however, are not antisymmetric but must have conjugate symmetry in the sense that rows and columns are interchanged in the respective Young tableaux, indicated by $[f_1f_2f_3f_4]$ and $[f_1'f_2'f_3'f_4']$.

To further clarify the meaning the $\mathrm{SU}_{TS}(4)$ labels it is instructive to analyse the two-particle case, the classification of which is summarised in Table 1 for the *sd* shell. Total antisymmetry of the wavefunction ensures in this case that the spatial part is symmetric and the isospin–spin part antisymmetric ($[f_1f_2f_3f_4] = [1100]$) or the spatial part is antisymmetric and the isospin–spin part symmetric ($[f_1f_2f_3f_4] = [2000]$), with the corresponding values of L, T and S as given in the table. This can be generalised to an arbitrary number of particles and the result emerges that the $\mathrm{SU}_{TS}(4)$ quantum numbers $(\lambda\mu\nu)$ specify the way in which the overall antisymmetry is distributed over the spatial and isospin–spin parts of the wavefunction.

The relevance of the classification (33) is its connection with the short-range nature of the residual interaction as a result of which states with spatial symmetry are favoured energetically. To see this point one may return once more to the example of two particles and consider an extreme form of a short-range interaction, namely a delta interaction. It can be shown that (see, e.g., chapter 11 of Talmi's book[9])

$$\langle (200) L M_L T M_T S M_S | \delta(\vec{r}_1 - \vec{r}_2 | (200) L M_L T M_T S M_S \rangle = 0, \qquad (34)$$

that is, the interaction matrix element vanishes identically for $(\lambda\mu\nu) = (200)$. Intuitively, this must be so since a spatially antisymmetric two-particle state has zero probability of having $\vec{r}_1 = \vec{r}_2$. In contrast, the matrix element is attractive in the orbitally symmetric case $(\lambda\mu\nu) = (010)$. Again, this result can be generalised to many particles, leading to the conclusion that the energy of a state depends strongly on $(\lambda\mu\nu)$. This statement can be quantified (see chapter 29 of Talmi's book[9]) by constructing a space exchange operator [related to the Casimir operator of $SU_{TS}(4)$] that measures the symmetry of the spatial part of the wavefunction.

The summary of the situation is as follows. The short-range character of the residual interaction favours spatial symmetry and, because of the connection between the space exchange operator and the Casimir operator of SU(4), this feature induces an energy splitting between the different SU(4) representations $(\lambda\mu\nu)$. It should be emphasised, however, that the $\vec{l}_k \cdot \vec{s}_k$ term in the mean-field potential (13) does not satisfy the second and third commutation relation in (29). The spin–orbit term breaks SU(4) symmetry [in a non-dynamical manner in the sense that SU(4) representations are admixed by it] and does so increasingly in heavier nuclei, since the energy splitting between the spin doublets $l - \frac{1}{2}$ and $l + \frac{1}{2}$ increases with A. In addition, SU(4) symmetry is also broken by the Coulomb interaction [which has a non-vanishing first and third commutator in (29)], an effect that also increases with nuclear mass, and might be broken by a spin dependence of the residual interaction. The result of these combined effects is that SU(4) is a good symmetry in light nuclei but fades away with increasing mass.

No statement is made in (33) about the nature of the orbital classification except that the total orbital angular momentum is assumed to be a good quantum number; additional labels are left unspecified as indicated by the dots. As discussed in the next section, the main feature of Elliott's model is that it provides an orbital classification scheme based on SU(3) which incorporates rotational characteristics.

3.4 Elliott's SU(3) Model of Rotation

Elliott's SU(3) model of rotation[15] presupposes Wigner's SU(4) classification and assumes in addition that the residual interaction has a ***quadrupole*** character. The latter is a reasonable hypothesis if both neutrons and protons are filling the same valence shell. In terms of the schematic hamiltonian (14) one requires that it reduces to

$$\hat{H}_{\rm SM} = \sum_{k=1}^{A} \left(\frac{p_k^2}{2m} + \tfrac{1}{2} m\omega^2 r_k^2 \right) - \chi \hat{Q} \cdot \hat{Q}, \tag{35}$$

where the quadrupole operator $\hat{Q}_\mu$ is given by

$$\hat{Q}_\mu = \sqrt{\frac{4\pi}{5}} \left(\sum_k r_k^2 Y_{2\mu}(\hat{r}_k)/b^2 + b^2 \sum_k p_k^2 Y_{2\mu}(\hat{p}_k)/\hbar^2 \right), \tag{36}$$

in terms of coordinates $\vec{r}_k$ and momenta $\vec{p}_k$, and the associated directional vectors $\hat{r}_k$ and $\hat{p}_k$.

To see that the shell-model hamiltonian (35) is analytically solvable, the starting point is to consider the commutation relations among the quadrupole operators (36) and the angular momentum operators

$$\hat{L}_\mu = \sum_k (r_k \wedge p_k)_\mu / \hbar. \tag{37}$$

The relevant commutators are

$$\begin{aligned} [\hat{L}_\mu, \hat{L}_\nu] &= -\sqrt{2} \langle 1\mu\ 1\nu | 1\mu+\nu \rangle \hat{L}_{\mu+\nu}, \\ [\hat{Q}_\mu, \hat{L}_\nu] &= -\sqrt{6} \langle 2\mu\ 1\nu | 2\mu+\nu \rangle \hat{Q}_{\mu+\nu}, \\ [\hat{Q}_\mu, \hat{Q}_\nu] &= 3\sqrt{10} \langle 2\mu\ 2\nu | 1\mu+\nu \rangle \hat{L}_{\mu+\nu}, \end{aligned} \tag{38}$$

and they show that the 9 operators $\hat{L}_\mu$ and $\hat{Q}_\mu$ form an SU(3) Lie algebra, here denoted as $\mathrm{SU}_L(3)$. The number of oscillator quanta is given by the one-body term in (35), which thus can be interpreted as a number operator,

$$\hbar\omega \left(\hat{n} + \tfrac{3}{2} \right) = \sum_{k=1}^{A} \left(\frac{p_k^2}{2m} + \tfrac{1}{2} m\omega^2 r_k^2 \right). \tag{39}$$

The 10 operators $\hat{n}$, $\hat{L}_\mu$ and $\hat{Q}_\mu$ together form the Lie algebra U(3). On the basis of the commutation relations (38) it can be shown that the quadratic Casimir operator of $\mathrm{SU}_L(3)$ is

$$\hat{C}_2[\mathrm{SU}_L(3)] = \tfrac{1}{4} \hat{Q} \cdot \hat{Q} + \tfrac{3}{4} \hat{L} \cdot \hat{L}, \tag{40}$$

and that the quadrupole interaction can thus be written as

$$\hat{Q}\cdot\hat{Q} = 4\hat{C}_2[\mathrm{SU}_L(3)] - 3\hat{C}_2[\mathrm{O}_L(3)]. \tag{41}$$

This shows that the shell-model hamiltonian (35) can be rewritten as

$$\hat{H}_{\mathrm{SM}} = \hbar\omega\left(\hat{n}+\tfrac{3}{2}\right) - \chi\left(4\hat{C}_2[\mathrm{SU}_L(3)] - 3\hat{C}_2[\mathrm{O}_L(3)]\right), \tag{42}$$

and that its eigensolutions are classified according to

$$\begin{array}{ccccccccc}
\mathrm{U}(4\omega) & \supset & \left(\mathrm{U}_L(\omega)\right. & \supset & \mathrm{U}_L(3) & \supset & \mathrm{SU}_L(3) & \supset & \left.\mathrm{O}_L(3)\right) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
[1^n] & & [f_1'f_2'f_3'f_4'] & & [\bar{f}_1\bar{f}_2\bar{f}_3] & & (\bar{\lambda}\bar{\mu}) & & KL \\
\otimes & & \left(\mathrm{U}_{TS}(4)\right. & \supset & \mathrm{SU}_{TS}(4) & \supset & \mathrm{SU}_T(2) & \otimes & \left.\mathrm{SU}_S(2)\right) \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & [f_1f_2f_3f_4] & & (\lambda\mu\nu) & & T & & S
\end{array}, \tag{43}$$

where the $\mathrm{SU}_L(3)$ labels are defined according to Elliott's convention [15]: $\bar{\lambda} = \bar{f}_1 - \bar{f}_2$ and $\bar{\mu} = \bar{f}_2 - \bar{f}_3$. The eigenstates are labelled by

$$|[1^n](\lambda\mu\nu)(\bar{\lambda}\bar{\mu})KLSJM_JTM_T\rangle, \tag{44}$$

where superfluous labels are omitted. The hamiltonian (35) only acts in orbital space; given a Wigner supermultiplet representation $(\lambda\mu\nu)$, it thus provides a splitting within that space which is of the form

$$\begin{aligned}
&\left[\sum_{k=1}^{A}\left(\frac{p_k^2}{2m}+\tfrac{1}{2}m\omega^2r_k^2\right) - \chi\hat{Q}\cdot\hat{Q}\right]|[1^n](\lambda\mu\nu)(\bar{\lambda}\bar{\mu})KLSJM_JTM_T\rangle \\
&= \left[\hbar\omega\left(\bar{f}_1+\bar{f}_2+\bar{f}_3+\tfrac{3}{2}\right) - 4\chi\left(\bar{\lambda}^2+\bar{\mu}^2+3(\bar{\lambda}+\bar{\mu})+\bar{\lambda}\bar{\mu}\right) + 3\chi L(L+1)\right] \\
&\quad\times|[1^n](\lambda\mu\nu)(\bar{\lambda}\bar{\mu})KLSJM_JTM_T\rangle.
\end{aligned} \tag{45}$$

The summary of the situation is as follows. The starting point of Elliott's SU(3) model is Wigner's supermultiplet scheme which proposes a classification of states based on their isospin–spin symmetry character as a result of the short-range nature of the residual interaction. On top of that, Elliott proposes an SU(3) classification of the orbital part of the wavefunction assuming a residual interaction of quadrupole type.

The importance of Elliott's idea is that it gives rise to a rotational classification of states in the context of the shell model through mixing of spherical

orbits. The quantum number K emerging from his analysis is associated with the projection of total orbital angular momentum on the axis of symmetry. As such the SU(3) model established a link between the nuclear shell model of Mayer [16] and of Jensen *et al.* [17] and the droplet model of Bohr and Mottelson [18] which up to then (1958) existed as two separate views of the nucleus. It is telling that this link was established through an analytically solvable, algebraic model rather than via some cumbersome numerical diagonalisation.

Given that Elliott's SU(3) model uses Wigner's supermultiplet classification as a starting point, it likewise breaks down as a result of the spin–orbit term in the nuclear mean field and it cannot be applied to heavy nuclei. Since in the s and p shells the SU(3) model reduces to Wigner's supermultiplet theory, the first real test case where orbital mixing and associated deformation may occur, is for nuclei in the *sd* shell. Elliott's SU(3) model has thus found its main application in *sd*-shell nuclei since beyond ^{40}Ca the SU(4) approximation breaks down due to the spin–orbit coupling.

4 Conclusion

It has not been my intention here to give a comprehensive overview of the application of symmetry in nuclear models. Rather, I wanted to sketch what I believe to be the historically most important developments in this field and do so from a unified perspective. Because of the latter feature, the formulation followed here differs from the ones adopted by Wigner, Racah and Elliott but the physical ideas at the basis of their respective models remains intact.

The fact that I have given here preponderance to the older models of Wigner, Racah and Elliott does not mean that there are no novel developments in this field. Elliott's SU(3) model has been extended to heavier nuclei by Ratna Raju, Draayer and collaborators [19], following early ideas of Hecht and Adler [20] and Arima *et al.* [21] leading to what is now known as the pseudo-SU(3) model, used actively in the interpretation of a wide range of nuclear data. The SU(3) model which considers orbital mixing within a single harmonic-oscillator shell, has also been extended to include $2\hbar\omega$ mixing by Rosensteel and Rowe [22]. Another development arose in the context of the fermion dynamical symmetry model [23] which admits a wide class of analytically solvable fermion hamiltonians, following original ideas of Ginocchio [24].

Arguably the most important development in the application of algebraic methods to nuclear structure has been the interacting boson model of Arima and Iachello [25]. Its impact can probably be understood as a result of two of its features: the elegance and richness of its algebraic structure and its ease of use in the interpretation of many data of virtually all but the very

lightest nuclei. Although I chose not to discuss here the symmetries of the interacting boson model for reasons explained in the introduction, I am fully aware of its importance. While Arima and Iachello borrowed several techniques from Racah and Elliott, it was they who pioneered the idea that the algebraic approach can be used for a comprehensive description of the structure of nuclei.

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References

1. P. Van Isacker, in preparation for Reports on Progress in Physics.
2. J.P. Elliott and P.G. Dawber, *Symmetry in Physics. Volumes 1 & 2* (Oxford University Press, Oxford, 1979).
3. A. Frank and P. Van Isacker, *Algebraic Methods in Molecular and Nuclear Physics* (Wiley-Interscience, New York, 1994).
4. F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge University Press, Cambridge, 1987).
5. F. Iachello and P. Van Isacker, *The Interacting Boson–Fermion Model* (Cambridge University Press, Cambridge, 1991).
6. P.J. Brussaard and P.W.M. Glaudemans, *Shell-Model Applications in Nuclear Spectroscopy* (North-Holland, Amsterdam, 1977).
7. R.D. Lawson, *Theory of the Nuclear Shell Model* (Oxford University Press, Oxford, 1980).
8. K.L.G. Heyde, *The Nuclear Shell Model* (Springer–Verlag, Berlin, 1990).
9. I. Talmi, *Simple Models of Complex Nuclei. The Shell Model and the Interacting Boson Model* (Harwood, New York, 1993).
10. A.K. Kerman, Ann. Phys. (NY) **12** (1961) 300.
11. G. Racah, Phys. Rev. **63** (1943) 367.
12. W. Heisenberg, Z. Phys. **77** (1932) 1.
13. E.P. Wigner, Phys. Rev. **51** (1937) 106.
14. F. Hund, Z. Phys. **105** (1937) 202.
15. J.P. Elliott, Proc. Roy. Soc. (London) A **245** (1958) 128; 562.

16. M.G. Mayer, Phys. Rev. **75** (1949) 1969.
17. J.H.D. Jensen, H. Suess and O. Haxel, Die Naturwissenschaften **36** (1949) 155.
18. A. Bohr and B.R. Mottelson, Dan. Vid. Selsk. Mat.-Fys. Medd. No. 16 (1953).
19. R.D. Ratna Raju, J.P. Draayer and K.T. Hecht, Nucl. Phys. A **202** (1973) 433.
20. K.T. Hecht and A. Adler, Nucl. Phys. A **137** (1969) 129.
21. A. Arima, M. Harvey and K. Shimizu, Phys. Lett. B **30** (1969) 517.
22. G. Rosensteel and D.J. Rowe, Phys. Rev. Lett. **38** (1977) 10; Ann. Phys. (NY) **126** (1980) 343.
23. C.L. Wu, D.H. Feng, X.G. Chen, J.Q. Chen and M.W. Guidry, Phys. Lett. B **168** (1986) 313.
24. J.N. Ginocchio, Ann. Phys. (NY) **126** (1980) 234.
25. A. Arima and F. Iachello, Ann. Phys. (NY) **99** (1976) 253; **111** (1978) 201; **123** (1979) 468.