d=2 N=2 superconformal symmetries and models

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$d = 2,\ N = 2$
Superconformal Symmetries and Models

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We discuss the following aspects of two-dimensional $N = 2$ supersymmetric theories defined on compact super Riemann surfaces: parametrization of $(2, 0)$ and $(2, 2)$ superconformal structures in terms of Beltrami coefficients and formulation of superconformal models on such surfaces (invariant actions, anomalies and compensating actions, Ward identities).

Chapter 1

Introduction

The reasons for studying 2-dimensional $N = 2$ superconformal field theories are numerous and well known (e.g. see [1]): the areas of application include string theory, mirror symmetry, topological field theories, exactly solvable models, quantum and W-gravity. Since holomorphic factorization represents a fundamental property of many of these models [2], it is particularly interesting to have a field theoretic approach in which holomorphic factorization is realized in a manifest way by virtue of an appropriate parametrization of the basic variables.

The goal of the present work is to develop such an approach to the superspace formulation of (2,2) and (2,0) superconformal models. In order to describe this approach and its relationship to other formulations in more detail, it is useful to summarize briefly previous work in this field.

The $d = 2, N = 2$ superconformally invariant coupling of matter fields to gravity was first discussed in the context of the fermionic string [3, 4]. Later on, the analogous (2,0) supersymmetric theory has been introduced and sigma-model couplings have been investigated [5, 6, 7]. Some of this work has been done in component field formalism, some other in superspace formalism. The latter has the advantage that *supersymmetry is manifestly realized* and that *field-dependent symmetry algebras are avoided*. (Such algebras usually occur in the component field formalism (WZ-gauge) [8].)

The geometry of $d = 2, N = 2$ superspace and the classification of irreducible multiplets has been analyzed by the authors of references [9, 10, 11, 12]. As is well known [13, 14], the quantization of supergravity in superspace requires the explicit solution of the constraints imposed on the geometry in terms of prepotential superfields. In two dimensions, these prepotentials (parametrizing superconformal classes of metrics) represent superspace expressions of the Beltrami differentials [15]. The determination of an explicit solution for the (2,0) and (2,2) constraints has been studied in references [16, 17, 18, 19] and [20, 21, 22], respectively.

On the other hand, a field theoretic approach to (ordinary) conformal models in which holomorphic factorization is manifestly realized was initiated by R.Stora and developed by several authors [23, 24]. This formalism comes in two versions.

One may formulate the theory on a Riemannian manifold in which case one has to deal with Weyl rescalings of the metric and with conformal classes of metrics parametrized by Beltrami coefficients. Alternatively, one may work on a Riemann surface in which case one simply deals with complex structures which are equivalent to conformal classes of metrics. This Riemannian surface approach enjoys the following properties. *Locality* is properly taken into account, *holomorphic factorization* is realized manifestly due to a judicious choice of variables and the theory is *globally defined* on a compact Riemann surface of arbitrary genus. Furthermore, the fact of working right away on a Riemann surface (i.e. with a conformal class of metrics) renders this approach more *economical* since there is no need for introducing Weyl rescalings and eliminating these degrees of freedom in the sequel.

The Riemannian manifold approach [24] has been generalized to the $N = 1$ supersymmetric case in reference [25] and to the $(2, 2)$ and $(2, 0)$ supersymmetric cases in references [21] and [18], respectively. The Riemannian surface approach [23] has been extended to the $N = 1$ supersymmetric theory in reference [26] and was used to prove the superholomorphic factorization theorem for partition functions on Riemann surfaces [27]. Both of these approaches to superconformal models are formulated in terms of Beltrami superfields ('prepotentials') and their relationship with the usual (Siegel-Gates like) solution of supergravity constraints has been discussed in references [26] and [15]. We will come back to this issue in the concluding section where we also mention further applications. It should be noted that the generalization to $N = 2$ supersymmetry is more subtle than the one to the $N = 1$ theory due to the appearance of an extra U(1)-symmetry.

Our paper is organized as follows. We first consider the (2,0) theory since it allows for simpler notation and calculations. Many results for the z-sector of the (2,0) theory have the same form as those of the z-sector of the (2,2) theory (the corresponding results for the $\bar{z}$-sector being obtained by complex conjugation). After a detailed presentation of the (2,0) theory, we simply summarize the results for the (2,2) theory. Comparison of our results with those of other approaches will be made within the text and in the concluding section.

Chapter 2

$N = 2$ Superconformal symmetry

In this chapter, we introduce $N = 2$ superconformal transformations and some related notions [28, 29, 30, 6, 14]. To keep supersymmetry manifest, all considerations will be carried out in superspace [31, 13, 14, 8], but the projection of the results to ordinary space will be outlined in the end.

2.1 Superconformal transformations and SRS's

Notation and basic relations

An $N = 2$ super Riemann surface (SRS) is locally parametrized by coordinates

$$(\mathcal{Z};\bar{\mathcal{Z}}) \equiv (z,\theta,\bar{\theta};\bar{z},\theta^-,\bar{\theta}^-) \equiv (x^{++},\theta^+,\bar{\theta}^+;x^{--},\theta^-,\bar{\theta}^-) \quad , \tag{2.1}$$

with $z, \bar{z}$ even and $\theta, \bar{\theta}, \theta^-, \bar{\theta}^-$ odd. The variables are complex and related by complex conjugation (denoted by $*$):

$$z^* = \bar{z} \qquad , \qquad (\theta^+)^* = \theta^- \qquad , \qquad (\bar{\theta}^+)^* = \bar{\theta}^- \quad .$$

As indicated in (2.1), we will omit the plus-indices of θ^+ and $\bar{\theta}^+$ to simplify the notation.

The canonical basis of the tangent space is defined by (∂, D, $\bar{D}$; $\bar{\partial}$, D_-, $\bar{D}_-$) with

$$\begin{array}{rclcrclcrcl} \partial &=& \dfrac{\partial}{\partial z} &,& D &=& \dfrac{\partial}{\partial \theta} + \dfrac{1}{2}\bar{\theta}\partial &,& \bar{D} &=& \dfrac{\partial}{\partial \bar{\theta}} + \dfrac{1}{2}\theta\partial \\ \bar{\partial} &=& \dfrac{\partial}{\partial \bar{z}} &,& D_- &=& \dfrac{\partial}{\partial \theta^-} + \dfrac{1}{2}\bar{\theta}^-\bar{\partial} &,& \bar{D}_- &=& \dfrac{\partial}{\partial \bar{\theta}^-} + \dfrac{1}{2}\theta^-\bar{\partial} \quad . \end{array} \tag{2.2}$$

The graded Lie brackets between these vector fields are given by

$$\{D, \bar{D}\} = \partial \qquad , \qquad \{D_-, \bar{D}_-\} = \bar{\partial} \quad , \tag{2.3}$$

all others brackets being zero, in particular,

$$D^2 = 0 = \bar{D}^2 \qquad , \qquad (D_-)^2 = 0 = (\bar{D}_-)^2 \quad . \tag{2.4}$$

For later reference, we note that this set of equations implies

$$[D, \bar{D}]^2 = \partial^2 \qquad , \qquad [D_-, \bar{D}_-]^2 = \bar{\partial}^2 \quad . \tag{2.5}$$

The cotangent vectors which are dual to the canonical tangent vectors (2.2) are given by the 1-forms

$$\begin{aligned} e^z &= dz + \frac{1}{2}\,\theta d\bar{\theta} + \frac{1}{2}\,\bar{\theta} d\theta \qquad , \quad e^\theta = d\theta \quad , \quad e^{\bar{\theta}} = d\bar{\theta} \\ e^{\bar{z}} &= d\bar{z} + \frac{1}{2}\,\theta^- d\bar{\theta}^- + \frac{1}{2}\,\bar{\theta}^- d\theta^- \quad , \quad e^{\theta^-} = d\theta^- \quad , \quad e^{\bar{\theta}^-} = d\bar{\theta}^- \end{aligned} \tag{2.6}$$

and that the graded commutation relations (2.3)(2.4) are equivalent to the *structure equations*

$$\begin{aligned} 0 &= de^z + e^\theta\, e^{\bar{\theta}} \qquad , \qquad de^\theta = 0 = de^{\bar{\theta}} \\ 0 &= de^{\bar{z}} + e^{\theta^-}\, e^{\bar{\theta}^-} \qquad , \qquad de^{\theta^-} = 0 = de^{\bar{\theta}^-} \quad . \end{aligned} \tag{2.7}$$

Superconformal transformations

By definition of the SRS, any two sets of local coordinates, say $(\mathcal{Z}; \bar{\mathcal{Z}})$ and $(\mathcal{Z}'; \bar{\mathcal{Z}}')$, are related by a superconformal transformation, i.e. a mapping for which D, $\bar{D}$ transform among themselves and similarly D_-, $\bar{D}_-$:

$$\begin{aligned} D &= [\,D\theta'\,]\,D' + [\,D\bar{\theta}'\,]\,\bar{D}' \quad , \quad D_- = [\,D_-\theta^{-\prime}\,]\,D'_- + [\,D_-\bar{\theta}^{-\prime}\,]\,\bar{D}'_- \\ \bar{D} &= [\,\bar{D}\theta'\,]\,D' + [\,\bar{D}\bar{\theta}'\,]\,\bar{D}' \quad , \quad \bar{D}_- = [\,\bar{D}_-\theta^{-\prime}\,]\,D'_- + [\,\bar{D}_-\bar{\theta}^{-\prime}\,]\,\bar{D}'_- \quad . \end{aligned} \tag{2.8}$$

These properties are equivalent to the following two conditions :
(i)

$$\begin{aligned} \mathcal{Z}' &= \mathcal{Z}'(\mathcal{Z}) \quad \Longleftrightarrow \quad D_-\mathcal{Z}' = 0 = \bar{D}_-\mathcal{Z}' \\ \bar{\mathcal{Z}}' &= \bar{\mathcal{Z}}'(\bar{\mathcal{Z}}) \quad \Longleftrightarrow \quad D\bar{\mathcal{Z}}' = 0 = \bar{D}\bar{\mathcal{Z}}' \quad , \end{aligned} \tag{2.9}$$

(ii)

$$\begin{aligned} Dz' &= \frac{1}{2}\theta'(D\bar{\theta}') + \frac{1}{2}\bar{\theta}'(D\theta') \qquad , \quad \bar{D}z' = \frac{1}{2}\theta'(\bar{D}\bar{\theta}') + \frac{1}{2}\bar{\theta}'(\bar{D}\theta') \\ D_-\bar{z}' &= \frac{1}{2}\theta^{-\prime}(D_-\bar{\theta}^{-\prime}) + \frac{1}{2}\bar{\theta}^{-\prime}(D_-\theta^{-\prime}) \ , \ \bar{D}_-\bar{z}' = \frac{1}{2}\theta^{-\prime}(\bar{D}_-\bar{\theta}^{-\prime}) + \frac{1}{2}\bar{\theta}^{-\prime}(\bar{D}_-\theta^{-\prime}). \end{aligned} \tag{2.10}$$

Application of the algebra (2.3)(2.4) to eqs.(2.10) yields a set of integrability conditions,

$$\begin{aligned} 0 &= (D\theta'\,)\,(D\bar{\theta}'\,) \\ 0 &= (\bar{D}\bar{\theta}'\,)\,(\bar{D}\theta'\,) \\ 0 &= (D\theta')\,(\bar{D}\bar{\theta}') + (D\bar{\theta}')\,(\bar{D}\theta') - \left[\partial z' + \frac{1}{2}\,\bar{\theta}'\,\partial\theta' + \frac{1}{2}\,\theta'\,\partial\bar{\theta}'\right] \end{aligned} \tag{2.11}$$

(and similarly for the $\bar{z}$-sector). Obviously, there are four possibilities to satisfy the first two of these equations. The two solutions $D\theta' = 0 = \bar{D}\theta'$ and $\bar{D}\bar{\theta}' = 0 = D\bar{\theta}'$ are not acceptable, because they would imply that the change of coordinates is non-invertible (the associated Berezinian would vanish). The third possibility, $D\theta' = 0 = \bar{D}\bar{\theta}'$ amounts to interchanging the rôle of θ and $\bar{\theta}$, since it leads to $D \propto \bar{D}'$ and $\bar{D} \propto D'$. The remaining solution is

$$D\bar{\theta}' \;=\; 0 \;= \bar{D}\theta' \quad , \tag{2.12}$$

which implies that D and $\bar{D}$ separately transform into themselves. The resulting transformation laws can be written as

$$\begin{aligned} D' &= \mathrm{e}^{w}\, D \\ \bar{D}' &= \mathrm{e}^{\bar{w}}\, \bar{D} \\ \partial' &= \{D', \bar{D}'\} = \mathrm{e}^{w+\bar{w}}\, [\partial + (\bar{D}w)D + (D\bar{w})\bar{D}] \end{aligned} \tag{2.13}$$

with

$$\begin{aligned} \mathrm{e}^{-w} &\equiv D\theta' \quad , \quad & Dw &= 0 \\ \mathrm{e}^{-\bar{w}} &\equiv \bar{D}\bar{\theta}' \quad , \quad & \bar{D}\bar{w} &= 0 \quad . \end{aligned} \tag{2.14}$$

The last equation in (2.11) then leads to

$$\mathrm{e}^{-w-\bar{w}} \;=\; \partial z' + \frac{1}{2}\,\bar{\theta}'\,\partial\theta' + \frac{1}{2}\,\theta'\,\partial\bar{\theta}' \quad . \tag{2.15}$$

In the remainder of the text, *superconformal transformations* are assumed to satisfy conditions (2.9)(2.10) and (2.12). Analogous equations hold in the $\bar{z}$-sector,

$$\begin{aligned} D'_- &= \mathrm{e}^{w^-} D_- \quad , \quad & \mathrm{e}^{-w^-} &\equiv D_-\theta^{-\prime} \quad , \quad & D_- w^- &= 0 \\ \bar{D}'_- &= \mathrm{e}^{\bar{w}^-} \bar{D}_- \quad , \quad & \mathrm{e}^{-\bar{w}^-} &\equiv \bar{D}_-\bar{\theta}^{-\prime} \quad , \quad & \bar{D}_-\bar{w}^- &= 0 \end{aligned} \tag{2.16}$$

with the relation

$$\mathrm{e}^{-w^- - \bar{w}^-} = \bar{\partial}\bar{z}' + \frac{1}{2}\bar{\theta}^{-\prime}\bar{\partial}\theta^{-\prime} + \frac{1}{2}\theta^{-\prime}\bar{\partial}\bar{\theta}^{-\prime} \quad . \tag{2.17}$$

To conclude our discussion, we note that the superconformal transformations of the canonical 1-forms read

$$\begin{aligned} e^{z'} &= \mathrm{e}^{-w-\bar{w}}\, e^{z} \quad , \quad & e^{\bar{z}'} &= \mathrm{e}^{-w^- - \bar{w}^-}\, e^{\bar{z}} \\ e^{\theta'} &= \mathrm{e}^{-w}\, [e^{\theta} - e^{z}(\bar{D}w)] \quad , \quad & e^{\theta^{-\prime}} &= \mathrm{e}^{-w^-}\, [e^{\theta^-} - e^{\bar{z}}(\bar{D}_- w^-)] \\ e^{\bar{\theta}'} &= \mathrm{e}^{-\bar{w}}\, [e^{\bar{\theta}} - e^{z}(D\bar{w})] \quad , \quad & e^{\bar{\theta}^{-\prime}} &= \mathrm{e}^{-\bar{w}^-}\, [e^{\bar{\theta}^-} - e^{\bar{z}}(D_-\bar{w}^-)] \end{aligned} \tag{2.18}$$

with $w, \bar{w}$ and $w^-, \bar{w}^-$ given by eqs.(2.14) and (2.16), respectively.

$U(1)$-symmetry and complex conjugation

The $N = 2$ supersymmetry algebra admits a $U(1) \otimes U(1)$ automorphism group. In the *Minkowskian framework*, the latter may be viewed as $SO(1,1) \otimes SO(1,1)$ in which case the Grassmannian coordinates $\theta, \bar{\theta}, \theta^-, \bar{\theta}^-$ are all real and independent or it may be regarded as $SO(2) \otimes SO(2)$ in which case the Grassmannian coordinates are complex and related by $\theta^* = \bar{\theta}$ and $(\theta^-)^* = \bar{\theta}^-$.

2.2 Projection to component fields

A generic $N = 2$ superfield admits the θ-expansion

$$\begin{aligned}
F(\mathcal{Z}\,;\bar{\mathcal{Z}}) \;&=\; a + \theta\alpha + \bar{\theta}\beta + \theta^-\gamma + \bar{\theta}^-\delta \\
&\quad +\theta\bar{\theta}b + \theta\theta^- c + \theta\bar{\theta}^- d + \bar{\theta}\theta^- e + \bar{\theta}\bar{\theta}^- f + \theta^-\bar{\theta}^- g \\
&\quad +\theta\bar{\theta}\theta^-\epsilon + \theta\bar{\theta}\bar{\theta}^-\zeta + \theta\theta^-\bar{\theta}^-\eta + \bar{\theta}\theta^-\bar{\theta}^-\lambda \\
&\quad +\theta\bar{\theta}\theta^-\bar{\theta}^- h \quad ,
\end{aligned} \tag{2.19}$$

where the component fields $a, \alpha, \beta, \ldots$ depend on z and $\bar{z}$. Equivalently, these space-time fields can be introduced by means of projection,

$$\begin{aligned}
F\,| \;&=\; a \\
DF\,| \;&=\; \alpha \quad , \quad \bar{D}F\,| = \beta \quad , \quad D_-F\,| = \gamma \quad , \quad \bar{D}_-F\,| = \delta \\
[D,\bar{D}]F\,| \;&=\; -2b \quad , \quad DD_-F\,| = -c \quad , \quad D\bar{D}_-F\,| = -d \\
\bar{D}D_-F\,| \;&=\; -e \quad , \quad \bar{D}\bar{D}_-F\,| = -f \quad , \quad [D_-,\bar{D}_-]F\,| = -2g \\
[D,\bar{D}]D_-F\,| \;&=\; -2\epsilon \quad , \quad [D,\bar{D}]\bar{D}_-F\,| = -2\zeta \\
D[D_-,\bar{D}_-]F\,| \;&=\; -2\eta \quad , \quad \bar{D}[D_-,\bar{D}_-]F\,| = -2\lambda \\
[D,\bar{D}][D_-,\bar{D}_-]F\,| \;&=\; 4h \quad ,
\end{aligned} \tag{2.20}$$

where the bar denotes the projection onto the lowest component of the corresponding superfield.

Chapter 3

(2,0) Theory

In this chapter, we discuss (2,0) SRS's and super Beltrami differentials. The projection of superspace results to ordinary space will be performed in the end.

3.1 (2,0) Super Riemann Surfaces

A $(2,0)$ SRS is locally parametrized by coordinates $(z,\bar{z},\theta,\bar{\theta})$, the notation being the same as the one for the $N=2$ theory discussed in the last chapter. The basic geometric quantities and relations are obtained from those of the $N=2$ theory by dropping the terms involving θ^- and $\bar{\theta}^-$. Thus, in the z-sector, one has the same equations as in the $N=2$ case. For later reference, we now summarize all relations which hold in the present case in terms of a generic system of coordinates $(Z,\bar{Z},\Theta,\bar{\Theta})$.

The canonical basis of the tangent space and of the cotangent space are respectively given by

$$\partial_Z = \frac{\partial}{\partial Z} \quad , \quad \partial_{\bar{Z}} = \frac{\partial}{\partial \bar{Z}} \quad , \quad D_\Theta = \frac{\partial}{\partial \Theta} + \frac{1}{2}\,\bar{\Theta}\partial_Z \quad , \quad D_{\bar{\Theta}} = \frac{\partial}{\partial \bar{\Theta}} + \frac{1}{2}\,\Theta\partial_Z \tag{3.1}$$

and

$$e^Z = dZ + \frac{1}{2}\,\Theta d\bar{\Theta} + \frac{1}{2}\,\bar{\Theta}d\Theta \quad , \quad e^{\bar{Z}} = d\bar{Z} \quad , \quad e^\Theta = d\Theta \quad , \quad e^{\bar{\Theta}} = d\bar{\Theta} \quad , \tag{3.2}$$

the *structure relations* having the form

$$\{D_\Theta, D_{\bar{\Theta}}\} = \partial_Z \qquad , \qquad (D_\Theta)^2 \; = 0 = (D_{\bar{\Theta}})^2 \qquad , \quad \ldots \tag{3.3}$$

and

$$0 = de^Z + e^\Theta e^{\bar{\Theta}} \qquad , \qquad 0 = de^{\bar{Z}} = de^\Theta = de^{\bar{\Theta}} \quad . \tag{3.4}$$

A change of coordinates $(Z,\bar{Z},\Theta,\bar{\Theta}) \to (Z',\bar{Z}',\Theta',\bar{\Theta}')$ is a *superconformal transformation* if it satisfies the conditions

$$Z' \;\; = \;\; Z'(Z,\Theta,\bar{\Theta}) \quad \Longleftrightarrow \quad 0 = \partial_{\bar{Z}} Z'$$

$$
\begin{array}{rclcl}
\Theta' & = & \Theta'(Z,\Theta,\bar{\Theta}) & \Longleftrightarrow & 0 = \partial_{\bar{Z}}\Theta' \\
\bar{\Theta}' & = & \bar{\Theta}'(Z,\Theta,\bar{\Theta}) & \Longleftrightarrow & 0 = \partial_{\bar{Z}}\bar{\Theta}' \\
\bar{Z}' & = & \bar{Z}'(\bar{Z}) & \Longleftrightarrow & 0 = D_{\Theta}\bar{Z}' = D_{\bar{\Theta}}\bar{Z}'
\end{array} \tag{3.5}
$$

and

$$
\begin{array}{rcl}
D_{\Theta} Z' & = & \frac{1}{2}\,\Theta'(D_{\Theta}\bar{\Theta}') + \frac{1}{2}\,\bar{\Theta}'(D_{\Theta}\Theta') \\
D_{\bar{\Theta}} Z' & = & \frac{1}{2}\,\Theta'(D_{\bar{\Theta}}\bar{\Theta}') + \frac{1}{2}\,\bar{\Theta}'(D_{\bar{\Theta}}\Theta') \quad ,
\end{array} \tag{3.6}
$$

as well as

$$
D_{\Theta}\bar{\Theta}' \;=\; 0 \;=\; D_{\bar{\Theta}}\Theta' \quad . \tag{3.7}
$$

The induced change of the canonical tangent and cotangent vectors reads

$$
\begin{array}{rclcrcl}
D'_{\Theta} & = & \mathrm{e}^{W}\, D_{\Theta} & , & \partial'_{Z} & = & \mathrm{e}^{W+\bar{W}}\,[\partial_Z + (D_{\bar{\Theta}}W)D_{\Theta} + (D_{\Theta}\bar{W})D_{\bar{\Theta}}] \\
D'_{\bar{\Theta}} & = & \mathrm{e}^{\bar{W}}\, D_{\bar{\Theta}} & , & \partial'_{\bar{Z}} & = & (\partial_{\bar{Z}}\bar{Z}')^{-1}\,\partial_{\bar{Z}}
\end{array} \tag{3.8}
$$

and

$$
\begin{array}{rclcrcl}
e^{Z'} & = & \mathrm{e}^{-W-\bar{W}}\, e^{Z} & , & e^{\Theta'} & = & \mathrm{e}^{-W}\,[e^{\Theta} - e^{Z}\,(D_{\bar{\Theta}}W)] \\
e^{\bar{Z}'} & = & (\partial_{\bar{Z}}\bar{Z}')\, e^{\bar{Z}} & , & e^{\bar{\Theta}'} & = & \mathrm{e}^{-\bar{W}}\,[e^{\bar{\Theta}} - e^{Z}\,(D_{\Theta}\bar{W})]
\end{array} \tag{3.9}
$$

with

$$
\begin{array}{rclcrcl}
\mathrm{e}^{-W} & \equiv & D_{\Theta}\Theta' & , & D_{\Theta}W & = & 0 \\
\mathrm{e}^{-\bar{W}} & \equiv & D_{\bar{\Theta}}\bar{\Theta}' & , & D_{\bar{\Theta}}\bar{W} & = & 0
\end{array} \tag{3.10}
$$

and

$$
\mathrm{e}^{-W-\bar{W}} = \partial_Z Z' + \frac{1}{2}\,\bar{\Theta}'\partial_Z\Theta' + \frac{1}{2}\,\Theta'\partial_Z\bar{\Theta}' \quad . \tag{3.11}
$$

In the Euclidean framework, Θ and $\bar{\Theta}$ are independent complex variables and the action functional will also represent a complex quantity. In the Minkowskian setting, one either deals with real independent coordinates Θ and $\bar{\Theta}$ ($SO(1,1)$ automorphism group) or with complex conjugate variables Θ and $\Theta^* = \bar{\Theta}$ ($SO(2)$ automorphism group).

3.2 Beltrami superfields and U(1)-symmetry

Beltrami (super)fields parametrize (super)conformal structures with respect to a given (super)conformal structure. Thus, we start from a reference complex structure corresponding to a certain choice of local coordinates $(z, \bar{z}, \theta, \bar{\theta})$ for which we denote the canonical tangent vectors by

$$
\partial = \frac{\partial}{\partial z} \quad , \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}} \quad , \quad D \equiv D_{\theta} = \frac{\partial}{\partial \theta} + \frac{1}{2}\,\bar{\theta}\partial \quad , \quad \bar{D} \equiv D_{\bar{\theta}} = \frac{\partial}{\partial \bar{\theta}} + \frac{1}{2}\,\theta\partial \quad .
$$

Then, we pass over to an arbitrary complex structure (corresponding to local coordinates $(Z, \bar{Z}, \Theta, \bar{\Theta})$) by a smooth change of coordinates

$$(z, \bar{z}, \theta, \bar{\theta}) \longrightarrow \Big(Z(z, \bar{z}, \theta, \bar{\theta}), \bar{Z}(z, \bar{z}, \theta, \bar{\theta}), \Theta(z, \bar{z}, \theta, \bar{\theta}), \bar{\Theta}(z, \bar{z}, \theta, \bar{\theta}) \Big) \quad . \tag{3.12}$$

To simplify the notation, we label the small coordinates by small indices a, b, e.g. $(e^a) = (e^z, e^{\bar{z}}, e^\theta, e^{\bar{\theta}})$, $(D_a) = (\partial, \bar{\partial}, D, \bar{D})$ and the capital coordinates by capital indices A, B.

The transformation of the canonical 1-forms induced by the change of coordinates (3.12) reads

$$e^B \;=\; \sum_{a=z,\bar{z},\theta,\bar{\theta}} e^a \, E_a{}^B \qquad \text{for} \quad B \;=\; Z, \bar{Z}, \Theta, \bar{\Theta} \quad .$$

Here, the $E_a{}^B$ are superfields whose explicit form is easy to determine from the expressions (3.2) and $d = e^a D_a$: for $a = z, \bar{z}, \theta, \bar{\theta}$, one finds

$$\begin{aligned} E_a{}^Z &= D_a Z - \frac{1}{2}(D_a \Theta)\bar{\Theta} - \frac{1}{2}(D_a \bar{\Theta})\Theta \\ E_a{}^\Theta &= D_a \Theta \quad , \quad E_a{}^{\bar{\Theta}} \;=\; D_a \bar{\Theta} \quad , \quad E_a{}^{\bar{Z}} \;=\; D_a \bar{Z} \quad . \end{aligned} \tag{3.13}$$

Since e^Z and $e^{\bar{Z}}$ transform homogeneously under the superconformal transformations (3.5)-(3.7), one can extract from them some Beltrami variables $H_a{}^b$ which are inert under these transformations: to do so, we factorize $E_z{}^Z$ and $E_{\bar{z}}{}^{\bar{Z}}$ in e^Z and $e^{\bar{Z}}$, respectively :

$$e^Z = [\, e^z + \sum_{a \neq z} e^a \, H_a{}^z \,] \, E_z{}^Z \quad , \quad e^{\bar{Z}} = [\, e^{\bar{z}} + \sum_{a \neq \bar{z}} e^a \, H_a{}^{\bar{z}} \,] \, E_{\bar{z}}{}^{\bar{Z}} \tag{3.14}$$

with

$$H_a{}^z \equiv \frac{E_a{}^Z}{E_z{}^Z} \quad \text{for } a = \bar{z}, \theta, \bar{\theta} \quad \text{and} \quad H_a{}^{\bar{z}} \equiv \frac{E_a{}^{\bar{Z}}}{E_{\bar{z}}{}^{\bar{Z}}} \quad \text{for } a = z, \theta, \bar{\theta} \, . \tag{3.15}$$

By construction, $E_a{}^Z$ and $E_a{}^{\bar{Z}}$ vary homogeneously under the transformations (3.5)-(3.7), in particular

$$E_z{}^{Z'} \;=\; \mathrm{e}^{-W-\bar{W}} \; E_z{}^Z \quad .$$

This transformation law and the index structure of $E_z{}^Z$ advises us to decompose this complex variable as

$$E_z{}^Z \;\equiv\; \Lambda_\theta{}^\Theta \, \bar{\Lambda}_{\bar{\theta}}{}^{\bar{\Theta}} \;\equiv\; \Lambda \, \bar{\Lambda} \tag{3.16}$$

with $\Lambda, \bar{\Lambda}$ transforming according to

$$\Lambda^{\Theta'} \;=\; \mathrm{e}^{-W} \, \Lambda^\Theta \quad , \quad \bar{\Lambda}^{\bar{\Theta}'} \;=\; \mathrm{e}^{-\bar{W}} \, \bar{\Lambda}^{\bar{\Theta}} \quad . \tag{3.17}$$

Then, we can use Λ and $\bar{\Lambda}$ to extract Beltrami coefficients from e^{Θ} and $e^{\bar{\Theta}}$, respectively, in analogy to $N=1$ supersymmetry [26] :

$$H_a{}^{\theta} \;=\; \frac{1}{\Lambda}\,[\,E_a{}^{\Theta} - H_a{}^{z}\,E_z{}^{\Theta}\,] \quad , \quad H_a{}^{\bar{\theta}} \;=\; \frac{1}{\bar{\Lambda}}\,[\,E_a{}^{\bar{\Theta}} - H_a{}^{z}\,E_z{}^{\bar{\Theta}}\,] \quad \text{for } a \;=\; \bar{z},\theta,\bar{\theta} \;. \tag{3.18}$$

The *final result* is best summarized in matrix form,

$$\Big(\, e^{Z} \;,\; e^{\bar{Z}} \;,\; e^{\Theta} \;,\; e^{\bar{\Theta}} \,\Big) \;=\; \Big(\, e^{z} \;,\; e^{\bar{z}} \;,\; e^{\theta} \;,\; e^{\bar{\theta}} \,\Big) \cdot M \cdot Q \tag{3.19}$$

with

$$M \;=\; \begin{pmatrix} 1 & H_z{}^{\bar{z}} & 0 & 0 \\ H_{\bar{z}}{}^{z} & 1 & H_{\bar{z}}{}^{\theta} & H_{\bar{z}}{}^{\bar{\theta}} \\ H_{\theta}{}^{z} & H_{\theta}{}^{\bar{z}} & H_{\theta}{}^{\theta} & H_{\theta}{}^{\bar{\theta}} \\ H_{\bar{\theta}}{}^{z} & H_{\bar{\theta}}{}^{\bar{z}} & H_{\bar{\theta}}{}^{\theta} & H_{\bar{\theta}}{}^{\bar{\theta}} \end{pmatrix} \quad , \quad Q \;=\; \begin{pmatrix} \Lambda\bar{\Lambda} & 0 & \tau & \bar{\tau} \\ 0 & \Omega & 0 & 0 \\ 0 & 0 & \Lambda & 0 \\ 0 & 0 & 0 & \bar{\Lambda} \end{pmatrix} \tag{3.20}$$

where

$$\Omega \equiv \Omega_{\bar{z}}{}^{\bar{Z}} \equiv E_{\bar{z}}{}^{\bar{Z}} \quad , \quad \tau \equiv \tau_z{}^{\Theta} \equiv E_z{}^{\Theta} \quad , \quad \bar{\tau} \equiv \bar{\tau}_z{}^{\bar{\Theta}} \equiv E_z{}^{\bar{\Theta}} \quad . \tag{3.21}$$

All the 'H' are invariant under the superconformal transformations (3.5)-(3.7). Under the latter, the factors $\Lambda, \bar{\Lambda}$ change according to eqs.(3.17) while Ω and $\tau, \bar{\tau}$ vary according to $\Omega^{\bar{Z}'} = \Omega^{\bar{Z}} \partial \bar{Z}' / \partial \bar{Z}$ and

$$\begin{aligned} \tau^{\Theta'} \;&=\; \mathrm{e}^{-W}\,[\,\tau^{\Theta} \;-\; \Lambda^{\Theta}\,\bar{\Lambda}^{\bar{\Theta}}\,(D_{\bar{\Theta}}W)\,] \\ \bar{\tau}^{\bar{\Theta}'} \;&=\; \mathrm{e}^{-\bar{W}}\,[\,\bar{\tau}^{\bar{\Theta}} \;-\; \Lambda^{\Theta}\,\bar{\Lambda}^{\bar{\Theta}}\,(D_{\Theta}\bar{W})\,] \quad . \end{aligned} \tag{3.22}$$

Obviously, the decomposition (3.16) has introduced a U(1)-symmetry which leaves $e^{Z}, e^{\bar{Z}}, e^{\Theta}, e^{\bar{\Theta}}$ invariant and which is given by

$$\begin{aligned} \Lambda' \;&=\; \mathrm{e}^{K}\,\Lambda \quad , & \bar{\Lambda}' \;&=\; \mathrm{e}^{-K}\,\bar{\Lambda} \\ (H_a{}^{\bar{\theta}})' \;&=\; \mathrm{e}^{K}\,H_a{}^{\bar{\theta}} \quad , & (H_a{}^{\theta})' \;&=\; \mathrm{e}^{-K}\,H_a{}^{\theta} \quad \text{for } a \;=\; \bar{z},\theta,\bar{\theta} \;, \end{aligned} \tag{3.23}$$

where K is an unconstrained superfield. In the sequel, we will encounter this symmetry in other places and forms.

Besides the transformations we have considered so far, there are the superconformal variations of the small coordinates under which the basis 1-forms change according to

$$\begin{aligned} e^{z'} \;&=\; \mathrm{e}^{-w-\bar{w}}\,e^{z} \quad , & e^{\theta'} \;&=\; \mathrm{e}^{-w}\,[\,e^{\theta} \;-\; e^{z}\,(\bar{D}w)\,] \\ e^{\bar{z}'} \;&=\; e^{\bar{z}}\,\bar{\partial}\bar{z}' \quad , & e^{\bar{\theta}'} \;&=\; \mathrm{e}^{-\bar{w}}\,[\,e^{\bar{\theta}} \;-\; e^{z}\,(D\bar{w})\,] \end{aligned} \tag{3.24}$$

with $Dw \;=\; 0 \;=\; \bar{D}\bar{w}$. The determination of the induced transformations of the 'H' and of $\Lambda, \bar{\Lambda}, \Omega, \tau, \bar{\tau}$ is straightforward and we only present the results to which we will refer later on. In terms of the quantity

$$Y = 1 + (\bar{D}w)\,H_{\theta}{}^{z} + (D\bar{w})H_{\bar{\theta}}{}^{z} \quad ,$$

the combined superconformal and $U(1)$ transformation laws have the form

$$
\begin{array}{rcl}
\Lambda' &=& e^{K}\, e^{w}\, Y^{1/2}\, \Lambda \qquad , \qquad \bar{\Lambda}' \;=\; e^{-K}\, e^{\bar{w}}\, Y^{1/2}\, \bar{\Lambda} \qquad , \qquad \Omega' \;=\; (\bar{\partial}\bar{z}')^{-1}\, \Omega \\
H_{\theta'}{}^{z'} &=& e^{-\bar{w}}\, Y^{-1}\, H_{\theta}{}^{z} \qquad , \qquad H_{\bar{\theta}'}{}^{z'} \;=\; e^{-w}\, Y^{-1}\, H_{\bar{\theta}}{}^{z} \\
H_{\theta'}{}^{\bar{\theta}'} &=& e^{+K}\, e^{+w-\bar{w}}\, Y^{-1/2}\, \Big\{ H_{\theta}{}^{\bar{\theta}} + Y^{-1} [\, (\bar{D}w)\, H_{\theta}{}^{\bar{\theta}} + (D\bar{w}) H_{\bar{\theta}}{}^{\bar{\theta}}] H_{\theta}{}^{z} \Big\} \\
H_{\bar{\theta}'}{}^{\theta'} &=& e^{-K}\, e^{-w+\bar{w}}\, Y^{-1/2}\, \Big\{ H_{\bar{\theta}}{}^{\theta} + Y^{-1} [\, (D\bar{w})\, H_{\bar{\theta}}{}^{\theta} + (\bar{D}w) H_{\theta}{}^{\theta}] H_{\bar{\theta}}{}^{z} \Big\} \\
H_{\bar{\theta}'}{}^{\bar{\theta}'} &=& e^{+K}\, Y^{-1/2}\, \Big\{ H_{\bar{\theta}}{}^{\bar{\theta}} + Y^{-1} [\, (D\bar{w})\, H_{\bar{\theta}}{}^{\bar{\theta}} + (\bar{D}w) H_{\theta}{}^{\bar{\theta}}] H_{\bar{\theta}}{}^{z} \Big\} \\
H_{\theta'}{}^{\theta'} &=& e^{-K}\, Y^{-1/2}\, \Big\{ H_{\theta}{}^{\theta} + Y^{-1} [\, (\bar{D}w)\, H_{\theta}{}^{\theta} + (D\bar{w}) H_{\bar{\theta}}{}^{\theta}] H_{\theta}{}^{z} \Big\} \\
H_{\bar{z}'}{}^{z'} &=& e^{-w-\bar{w}}\, (\bar{\partial}\bar{z}')^{-1}\, Y^{-1}\, H_{\bar{z}}{}^{z} \\
H_{\theta'}{}^{\bar{z}'} &=& e^{w}\, (\bar{\partial}\bar{z}')\, H_{\theta}{}^{\bar{z}} \qquad , \qquad H_{\bar{\theta}'}{}^{\bar{z}'} \;=\; e^{\bar{w}}\, (\bar{\partial}\bar{z}')\, H_{\bar{\theta}}{}^{\bar{z}} \\
H_{z'}{}^{\bar{z}'} &=& e^{w+\bar{w}}\, (\bar{\partial}\bar{z}')\, \Big[H_{z}{}^{\bar{z}} + (\bar{D}w) H_{\theta}{}^{\bar{z}} + (D\bar{w}) H_{\bar{\theta}}{}^{\bar{z}} \Big] \quad .
\end{array}
\tag{3.25}
$$

The given variations of $\Lambda, \bar{\Lambda}$ and $H_a{}^{\theta}, H_a{}^{\bar{\theta}}$ result from a symmetric splitting of the transformation law

$$
(\Lambda\bar{\Lambda})' = e^{w+\bar{w}} Y (\Lambda\bar{\Lambda}) \quad .
$$

The ambiguity involved in this decomposition is precisely the $U(1)$-symmetry (3.23):

$$
\Lambda' = e^{K} e^{w} Y^{1/2} \Lambda \quad , \quad \bar{\Lambda}' = e^{-K} e^{\bar{w}} Y^{1/2} \bar{\Lambda} \quad .
$$

Due to the structure relations (3.4), not all of the *super Beltrami coefficients* $H_a{}^b$ and of the *integrating factors* $\Lambda, \bar{\Lambda}, \Omega, \tau, \bar{\tau}$ are independent variables. For instance, the structure relation $0 = de^{\bar{Z}}$ is equivalent to the set of equations

$$
\begin{array}{rcll}
0 &=& (\, D_a \, - \, H_a{}^{\bar{z}}\, \bar{\partial} \, - \, \bar{\partial} H_a{}^{\bar{z}}\,)\, \Omega & \text{for} \quad a = z, \theta, \bar{\theta} \\
0 &=& D_a (H_z{}^{\bar{z}} \Omega) \; - \; \partial (H_a{}^{\bar{z}} \Omega) & \text{for} \quad a = \theta, \bar{\theta} \\
0 &=& D (H_{\theta}{}^{\bar{z}} \Omega) & \\
0 &=& \bar{D} (H_{\bar{\theta}}{}^{\bar{z}} \Omega) & \\
0 &=& \bar{D} (H_{\theta}{}^{\bar{z}} \Omega) \; + \; D (H_{\bar{\theta}}{}^{\bar{z}} \Omega) \; - \; H_z{}^{\bar{z}} \Omega \quad . &
\end{array}
\tag{3.26}
$$

The last equation can be solved for $H_z{}^{\bar{z}}$ and the two equations preceding it provide constraints for the fields $H_{\theta}{}^{\bar{z}}$, $H_{\bar{\theta}}{}^{\bar{z}}$.

In summary, by solving all resulting equations which are algebraic, we find the following result. In the $\bar{z}$-sector, there is one integrating factor (Ω) and two independent Beltrami superfields ($H_{\theta}{}^{\bar{z}}$ and $H_{\bar{\theta}}{}^{\bar{z}}$), each of which satisfies a constraint reducing the number of its independent component fields by a factor 1/2. In section 3.9, the constraints on $H_{\theta}{}^{\bar{z}}$ and $H_{\bar{\theta}}{}^{\bar{z}}$ will be explicitly solved in terms of 'prepotential' superfields $H^{\bar{z}}$ and $\hat{H}^{\bar{z}}$. In the z-sector, there are two integrating factors (Λ, $\bar{\Lambda}$) and four independent and unconstrained Beltrami variables ($H_{\bar{z}}{}^{z}$, $H_{\theta}{}^{z}$, $H_{\bar{\theta}}{}^{z}$ and a non-U(1)-invariant combination of $H_{\theta}{}^{\theta}$, $H_{\bar{\theta}}{}^{\bar{\theta}}$, e.g. $H_{\theta}{}^{\theta}/H_{\bar{\theta}}{}^{\bar{\theta}}$). The dependent Beltrami fields only depend on the others and *not* on the integrating factors. This is an important point, since the integrating factors represent non-local functionals of the 'H' by virtue of the differential equations that they satisfy, see below.

To be more explicit, in the z-sector, one finds

$$\begin{aligned}
H_{\bar\theta}{}^{\theta} H_{\bar\theta}{}^{\bar\theta} &= -(\bar D - H_{\bar\theta}{}^{z}\partial) H_{\bar\theta}{}^{z} \quad , \qquad H_{\theta}{}^{\bar\theta} H_{\theta}{}^{\theta} = -(D - H_{\theta}{}^{z}\partial) H_{\theta}{}^{z} \\
H_{\theta}{}^{\theta} H_{\bar\theta}{}^{\bar\theta} + H_{\bar\theta}{}^{\theta} H_{\theta}{}^{\bar\theta} &= 1 - (\bar D - H_{\bar\theta}{}^{z}\partial) H_{\theta}{}^{z} - (D - H_{\theta}{}^{z}\partial) H_{\bar\theta}{}^{z} \\
H_{\bar z}{}^{\theta} H_{\theta}{}^{\bar\theta} + H_{\bar z}{}^{\bar\theta} H_{\theta}{}^{\theta} &= (D - H_{\theta}{}^{z}\partial) H_{\bar z}{}^{z} - (\bar\partial - H_{\bar z}{}^{z}\partial) H_{\theta}{}^{z} \\
H_{\bar z}{}^{\theta} H_{\bar\theta}{}^{\bar\theta} + H_{\bar z}{}^{\bar\theta} H_{\bar\theta}{}^{\theta} &= (\bar D - H_{\bar\theta}{}^{z}\partial) H_{\bar z}{}^{z} - (\bar\partial - H_{\bar z}{}^{z}\partial) H_{\bar\theta}{}^{z}
\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}
\tau &= (H_{\theta}{}^{\theta} H_{\bar\theta}{}^{\bar\theta} + H_{\bar\theta}{}^{\theta} H_{\theta}{}^{\bar\theta})^{-1} \Big[(\bar D - H_{\bar\theta}{}^{z}\partial)(H_{\theta}{}^{\theta}\Lambda) + (D - H_{\theta}{}^{z}\partial)(H_{\bar\theta}{}^{\theta}\Lambda) \Big] \\
\bar\tau &= (H_{\theta}{}^{\theta} H_{\bar\theta}{}^{\bar\theta} + H_{\bar\theta}{}^{\theta} H_{\theta}{}^{\bar\theta})^{-1} \Big[(D - H_{\theta}{}^{z}\partial)(H_{\bar\theta}{}^{\bar\theta}\bar\Lambda) + (\bar D - H_{\bar\theta}{}^{z}\partial)(H_{\theta}{}^{\bar\theta}\bar\Lambda) \Big] \, .
\end{aligned} \tag{3.28}$$

The determination of the independent fields in the set of equations (3.27) is best done by linearizing the variables according to $H_{\theta}{}^{\theta} = 1 + h_{\theta}{}^{\theta}, H_{\bar\theta}{}^{\bar\theta} = 1 + h_{\bar\theta}{}^{\bar\theta}$ and $H_a{}^b = h_a{}^b$ otherwise. The conclusion is the one summarized above.

Let us complete our discussion of the z-sector. The first of the structure relations (3.4) yields, amongst others, the following differential equation:

$$0 = (\, D_a - H_a{}^{z}\partial \,)\, (\Lambda\bar\Lambda) - (\partial H_a{}^{z})\, \Lambda\bar\Lambda - H_a{}^{\bar\theta}\, \tau\, \bar\Lambda - H_a{}^{\theta}\, \Lambda\, \bar\tau \qquad \text{for } \ a = \bar z, \theta, \bar\theta. \tag{3.29}$$

We note that this equation also holds for $a = z$ if we write the generic elements of the Beltrami matrix M of equation (3.20) as $H_a{}^b$ so that $H_z{}^z = 1$ and $H_z{}^\theta = 0 = H_z{}^{\bar\theta}$. The previous relation can be decomposed in a symmetric way with respect to Λ and $\bar\Lambda$ which leads to the *integrating factor equations* (IFEQ's)

$$\begin{aligned}
0 &= (\, D_a - H_a{}^{z}\partial - \frac{1}{2}\, \partial H_a{}^{z} - V_a \,)\, \Lambda - H_a{}^{\bar\theta}\, \tau \\
0 &= (\, D_a - H_a{}^{z}\partial - \frac{1}{2}\, \partial H_a{}^{z} + V_a \,)\, \bar\Lambda - H_a{}^{\theta}\, \bar\tau \quad .
\end{aligned} \tag{3.30}$$

The latter decomposition introduces a vector field V_a (with $V_z = 0$) which is to be interpreted as a connection for the U(1)-symmetry due to its transformation law under U(1)-transformations (see next section). It should be noted that V_a is not an independent variable, rather it is determined in terms of the 'H' by the structure equations:

$$\begin{aligned}
V_\theta &= \frac{-1}{H_{\theta}{}^{\theta}}\, [D - H_{\theta}{}^{z}\partial + \frac{1}{2}\, (\partial H_{\theta}{}^{z})]\, H_{\theta}{}^{\theta} \\
V_{\bar\theta} &= \frac{1}{H_{\bar\theta}{}^{\bar\theta}}\, [\bar D - H_{\bar\theta}{}^{z}\partial + \frac{1}{2}\, (\partial H_{\bar\theta}{}^{z})]\, H_{\bar\theta}{}^{\bar\theta}
\end{aligned} \tag{3.31}$$

$$
\begin{aligned}
V_{\bar{z}} &= \frac{1}{H_\theta{}^\theta} \left\{ [D - H_\theta{}^z \partial + \frac{1}{2} (\partial H_\theta{}^z) + V_\theta] \, H_{\bar{z}}{}^\theta - [\bar{\partial} - H_{\bar{z}}{}^z \partial + \frac{1}{2} (\partial H_{\bar{z}}{}^z)] \, H_\theta{}^\theta \right\} \\
&= \frac{-1}{H_{\bar{\theta}}{}^{\bar{\theta}}} \left\{ [\bar{D} - H_{\bar{\theta}}{}^z \partial + \frac{1}{2} (\partial H_{\bar{\theta}}{}^z) - V_{\bar{\theta}}] \, H_{\bar{z}}{}^{\bar{\theta}} - [\bar{\partial} - H_{\bar{z}}{}^z \partial + \frac{1}{2} (\partial H_{\bar{z}}{}^z)] \, H_{\bar{\theta}}{}^{\bar{\theta}} \right\} .
\end{aligned}
$$

By virtue of the relations between the 'H', the previous expressions can be rewritten in various other ways, for instance

$$
\begin{aligned}
-H_{\bar{\theta}}{}^\theta \, V_{\bar{\theta}} &= [\bar{D} - H_{\bar{\theta}}{}^z \partial \, + \, \frac{1}{2} (\partial H_{\bar{\theta}}{}^z)] \, H_{\bar{\theta}}{}^\theta \\
H_\theta{}^{\bar{\theta}} \, V_\theta &= [D - H_\theta{}^z \partial \, + \, \frac{1}{2} (\partial H_\theta{}^z)] \, H_\theta{}^{\bar{\theta}} \quad .
\end{aligned}
\tag{3.32}
$$

This finishes our discussion of the z-sector.

In the $\bar{z}$-sector, we have

$$
H_z{}^{\bar{z}} \; = \; (\bar{D} \, - \, H_{\bar{\theta}}{}^{\bar{z}} \bar{\partial}) H_\theta{}^{\bar{z}} \, + \, (D \, - \, H_\theta{}^{\bar{z}} \bar{\partial}) H_{\bar{\theta}}{}^{\bar{z}} \quad , \tag{3.33}
$$

where $H_\theta{}^{\bar{z}}$ and $H_{\bar{\theta}}{}^{\bar{z}}$ satisfy the covariant chirality conditions

$$
(\, D - H_\theta{}^{\bar{z}} \bar{\partial} \,) \, H_\theta{}^{\bar{z}} \; = \; 0 \; = \; (\, \bar{D} - H_{\bar{\theta}}{}^{\bar{z}} \bar{\partial} \,) \, H_{\bar{\theta}}{}^{\bar{z}} \quad . \tag{3.34}
$$

The first condition simply relates the component fields of $H_\theta{}^{\bar{z}}$ among themselves and the second those of $H_{\bar{\theta}}{}^{\bar{z}}$. Thereby, each of these superfields contains one independent bosonic and fermionic space-time component.

The factor Ω satisfies the IFEQ's

$$
0 \; = \; (\, D_a \, - \, H_a{}^{\bar{z}} \, \bar{\partial} \, - \, \bar{\partial} H_a{}^{\bar{z}} \,) \, \Omega \qquad \text{for} \quad a = z, \theta, \bar{\theta} \quad , \tag{3.35}
$$

the equation for z being a consequence of the ones for θ and $\bar{\theta}$.

3.3 Symmetry transformations

To deduce the transformation laws of the basic fields under infinitesimal superdiffeomorphisms, we proceed as in the $N = 0$ and $N = 1$ theories [26]. In the course of this process, the U(1)-transformations manifest themselves in a natural way.

Thus, we start from the ghost vector field

$$
\Xi \cdot \partial \; \equiv \; \Xi^z(z, \bar{z}, \theta, \bar{\theta}) \, \partial \; + \; \Xi^{\bar{z}}(z, \bar{z}, \theta, \bar{\theta}) \, \bar{\partial} \; + \; \Xi^\theta(z, \bar{z}, \theta, \bar{\theta}) \, D \; + \; \Xi^{\bar{\theta}}(z, \bar{z}, \theta, \bar{\theta}) \, \bar{D} \quad ,
$$

which generates an infinitesimal change of the coordinates $(z, \bar{z}, \theta, \bar{\theta})$. Following C.Becchi [24, 23], we consider a reparametrization of the ghosts,

$$
\left(C^z \, , \, C^{\bar{z}} \, , \, C^\theta \, , \, C^{\bar{\theta}} \right) \; = \; \left(\Xi^z \, , \, \Xi^{\bar{z}} \, , \, \Xi^\theta \, , \, \Xi^{\bar{\theta}} \right) \cdot M \quad , \tag{3.36}
$$

where M denotes the Beltrami matrix introduced in equation (3.20). Explicitly,

$$
\begin{array}{rcl}
C^z &=& \Xi^z \;+\; \Xi^{\bar z}\, H_{\bar z}{}^z \;+\; \Xi^\theta\, H_\theta{}^z \;+\; \Xi^{\bar\theta}\, H_{\bar\theta}{}^z \\
C^{\bar z} &=& \Xi^{\bar z} \;+\; \Xi^z\, H_z{}^{\bar z} \;+\; \Xi^\theta\, H_\theta{}^{\bar z} \;+\; \Xi^{\bar\theta}\, H_{\bar\theta}{}^{\bar z} \\
C^\theta &=& \Xi^\theta\, H_\theta{}^\theta \;+\; \Xi^{\bar z}\, H_{\bar z}{}^\theta \;+\; \Xi^{\bar\theta}\, H_{\bar\theta}{}^\theta \\
C^{\bar\theta} &=& \Xi^{\bar\theta}\, H_{\bar\theta}{}^{\bar\theta} \;+\; \Xi^{\bar z}\, H_{\bar z}{}^{\bar\theta} \;+\; \Xi^\theta\, H_\theta{}^{\bar\theta} \quad .
\end{array}
\tag{3.37}
$$

We note that the U(1)-transformations of the 'H', eqs.(3.23), induce those of the 'C',

$$
(C^z)' = C^z \quad , \quad (C^{\bar z})' = C^{\bar z} \quad , \quad (C^\theta)' = \mathrm{e}^{-K}\, C^\theta \quad , \quad (C^{\bar\theta})' = \mathrm{e}^{K}\, C^{\bar\theta} \quad ,
$$

but, for the moment being, we will not consider this symmetry and restrict our attention to the superdiffeomorphisms.

Contraction of the basis 1-forms (3.19) along the vector field $\Xi \cdot \partial$ gives

$$
\begin{array}{rcl}
i_{\Xi\cdot\partial}(e^Z) &=& \left[\Xi^z + \Xi^{\bar z} H_{\bar z}{}^z + \Xi^\theta H_\theta{}^z + \Xi^{\bar\theta} H_{\bar\theta}{}^z \right] \Lambda_\theta{}^\Theta \bar\Lambda_{\bar\theta}{}^{\bar\Theta} \\
&=& C^z \Lambda_\theta{}^\Theta \bar\Lambda_{\bar\theta}{}^{\bar\Theta} \\
i_{\Xi\cdot\partial}(e^\Theta) &=& \left[\Xi^z + \Xi^{\bar z} H_{\bar z}{}^z + \Xi^\theta H_\theta{}^z + \Xi^{\bar\theta} H_{\bar\theta}{}^z \right] \tau_z{}^\Theta + \left[\Xi^\theta H_\theta{}^\theta + \Xi^{\bar z} H_{\bar z}{}^\theta + \Xi^{\bar\theta} H_{\bar\theta}{}^\theta \right] \Lambda_\theta{}^\Theta \\
&=& C^z \tau_z{}^\Theta + C^\theta \Lambda_\theta{}^\Theta
\end{array}
\tag{3.38}
$$

and similarly

$$
i_{\Xi\cdot\partial}(e^{\bar\Theta}) \;=\; C^z\, \bar\tau_z{}^{\bar\Theta} \;+\; C^{\bar\theta}\, \bar\Lambda_{\bar\theta}{}^{\bar\Theta} \qquad , \qquad i_{\Xi\cdot\partial}(e^{\bar Z}) \;=\; C^{\bar z}\, \Omega_{\bar z}{}^{\bar Z} \quad .
$$

Thereby[1],

$$
\begin{array}{rcl}
s\Theta &=& i_{\Xi\cdot\partial}\, d\Theta \;=\; i_{\Xi\cdot\partial}\, e^\Theta \;=\; C^z \tau + C^\theta \Lambda \\
sZ &=& i_{\Xi\cdot\partial}\, dZ \;=\; i_{\Xi\cdot\partial}[\, e^Z - \frac{1}{2}\, \bar\Theta e^\Theta - \frac{1}{2}\, \Theta e^{\bar\Theta}\,] \;=\; C^z \Lambda\bar\Lambda - \frac{1}{2}\, \bar\Theta (s\Theta) - \frac{1}{2}\, \Theta (s\bar\Theta)
\end{array}
$$

and analogously

$$
s\bar\Theta \;=\; C^z\, \bar\tau \;+\; C^{\bar\theta}\, \bar\Lambda \qquad , \qquad s\bar Z \;=\; C^{\bar z}\, \Omega \quad .
$$

From the nilpotency of the s-operation, $0 = s^2 Z = s^2 \bar Z = s^2 \Theta = s^2 \bar\Theta$, we now deduce

$$
\begin{array}{rcl}
sC^z &=& -\, C^z\, (\Lambda\bar\Lambda)^{-1} \left[s(\Lambda\bar\Lambda) \;-\; C^{\bar\theta}\, \bar\Lambda\, \tau \;-\; C^\theta\, \Lambda\, \bar\tau \right] \;-\; C^\theta\, C^{\bar\theta} \\
sC^{\bar z} &=& -\, C^{\bar z}\, \Omega^{-1}\, [\, s\Omega\,] \\
sC^\theta &=& -\, \Lambda^{-1} \left[(sC^z)\, \tau \;+\; C^z\, (s\tau) \;+\; C^\theta\, (s\Lambda) \right] \\
sC^{\bar\theta} &=& -\, \bar\Lambda^{-1} \left[(sC^z)\, \bar\tau \;+\; C^z\, (s\bar\tau) \;+\; C^{\bar\theta}\, (s\bar\Lambda) \right] \quad .
\end{array}
\tag{3.39}
$$

[1] In superspace, the BRS-operator s is supposed to act as an antiderivation from the right and the ghost-number is added to the form degree, the Grassmann parity being s-inert [8].

The transformation laws of the integrating factors and Beltrami coefficients follow by evaluating in two different ways the variations of the differentials $dZ, d\bar{Z}, d\Theta, d\bar{\Theta}$; for instance[2],

$$s(d\Theta) \;=\; -d(s\Theta) \;=\; +[\, e^z\partial \;+\; e^{\bar{z}}\bar{\partial} \;+\; e^\theta D \;+\; e^{\bar{\theta}}\bar{D}\,]\,[\, C^z\,\tau \;+\; C^\theta\,\Lambda\,]$$

and

$$\begin{aligned} s(d\Theta) = se^\Theta \;&=\; \left[e^z + e^{\bar{z}}\, H_{\bar{z}}{}^z + e^\theta\, H_\theta{}^z + e^{\bar{\theta}}\, H_{\bar{\theta}}{}^z\right] s\tau + \left[e^{\bar{z}}\, sH_{\bar{z}}{}^z + e^\theta\, sH_\theta{}^z + e^{\bar{\theta}}\, sH_{\bar{\theta}}{}^z\right]\tau \\ &\quad + \left[e^\theta\, H_\theta{}^\theta + e^{\bar{z}}\, H_{\bar{z}}{}^\theta + e^{\bar{\theta}}\, H_{\bar{\theta}}{}^\theta\right] s\Lambda + \left[e^\theta sH_\theta{}^\theta + e^{\bar{z}} sH_{\bar{z}}{}^\theta + e^{\bar{\theta}} sH_{\bar{\theta}}{}^\theta\right]\Lambda \end{aligned}$$

lead to the variations of τ and $H_\theta{}^\theta, H_{\bar{z}}{}^\theta, H_{\bar{\theta}}{}^\theta$. More explicitly, comparison of the coefficients of e^z in both expressions for $s(d\Theta)$ yields

$$\begin{aligned} s\tau &= \partial\,(\, C^z\tau + C^\theta\Lambda\,) \\ s\bar{\tau} &= \partial\,(\, C^z\bar{\tau} + C^{\bar{\theta}}\bar{\Lambda}\,) \quad , \end{aligned} \tag{3.40}$$

where the second equation follows from $s(d\bar{\Theta})$ by the same lines of reasoning. From the coefficients of e^z in $s(dZ)$, one finds

$$s\,(\Lambda\bar{\Lambda}) \;=\; \partial\,(C^z\Lambda\bar{\Lambda}) \;+\; C^{\bar{\theta}}\,\bar{\Lambda}\,\tau \;+\; C^\theta\,\Lambda\,\bar{\tau} \quad . \tag{3.41}$$

In analogy to eqs.(3.29)(3.30), we decompose this variation in a symmetric way,

$$\begin{aligned} s\Lambda &= C^z\,\partial\Lambda \;+\; \frac{1}{2}\,(\partial C^z)\,\Lambda \;+\; C^{\bar{\theta}}\,\tau \;+\; K\,\Lambda \\ s\bar{\Lambda} &= C^z\,\partial\bar{\Lambda} \;+\; \frac{1}{2}\,(\partial C^z)\,\bar{\Lambda} \;+\; C^\theta\,\bar{\tau} \;-\; K\,\bar{\Lambda} \quad , \end{aligned} \tag{3.42}$$

where K denotes a ghost superfield. The K-terms which naturally appear in this decomposition represent an infinitesimal version of the U(1)-symmetry (3.23). The variation of the K-parameter follows from the requirement that the s-operator is nilpotent:

$$sK \;=\; -\left[\,C^z\partial K - \frac{1}{2}\,C^\theta(\partial C^{\bar{\theta}}) + \frac{1}{2}\,C^{\bar{\theta}}(\partial C^\theta)\,\right] \quad . \tag{3.43}$$

By substituting the expressions (3.40)-(3.42) into eqs.(3.39), we get

$$\begin{aligned} sC^z &= -\left[\,C^z\partial C^z + C^\theta C^{\bar{\theta}}\,\right] \\ sC^\theta &= -\left[\,C^z\partial C^\theta + \frac{1}{2}\,C^\theta(\partial C^z) - KC^\theta\,\right] \\ sC^{\bar{\theta}} &= -\left[\,C^z\partial C^{\bar{\theta}} + \frac{1}{2}\,C^{\bar{\theta}}(\partial C^z) + KC^{\bar{\theta}}\,\right] \quad . \end{aligned} \tag{3.44}$$

[2]For the action of the exterior differential d on ghost fields, see reference [8].

The variations of the Beltrami coefficients follow by taking into account the previous relations, the structure equations and eqs.(3.30) where the vector field V_a was introduced. They take the form

$$
\begin{aligned}
sH_a{}^z &= (\,D_a - H_a{}^z\partial + \partial H_a{}^z\,)\,C^z - H_a{}^\theta C^{\bar\theta} - H_a{}^{\bar\theta} C^\theta \\
sH_a{}^\theta &= (\,D_a - H_a{}^z\partial + \frac{1}{2}\,\partial H_a{}^z + V_a\,)\,C^\theta + C^z\partial H_a{}^\theta - \frac{1}{2}\,H_a{}^\theta(\partial C^z) - H_a{}^\theta K \\
sH_a{}^{\bar\theta} &= (\,D_a - H_a{}^z\partial + \frac{1}{2}\,\partial H_a{}^z - V_a\,)\,C^{\bar\theta} + C^z\partial H_a{}^{\bar\theta} - \frac{1}{2}\,H_a{}^{\bar\theta}(\partial C^z) + H_a{}^{\bar\theta} K.
\end{aligned}
\tag{3.45}
$$

Finally, the variation of V_a follows by requiring the nilpotency of the s-operations (3.45):

$$
sV_a = C^z\partial V_a + \frac{1}{2}\,H_a{}^\theta\partial C^{\bar\theta} - \frac{1}{2}\,(\partial H_a{}^\theta)C^{\bar\theta} - \frac{1}{2}\,H_a{}^{\bar\theta}\partial C^\theta + \frac{1}{2}\,(\partial H_a{}^{\bar\theta})C^\theta + (D_a - H_a{}^z\partial)K.
\tag{3.46}
$$

Equivalently, this transformation law can be deduced from the variations of the 'H' since V_a depends on these variables according to equations (3.31). The derivative of K in the variation (3.46) confirms the interpretation of V_a as a gauge field for the $U(1)$-symmetry.

In the $\bar z$-sector, the same procedure leads to the following results:

$$
\begin{aligned}
sH_a{}^{\bar z} &= (\,D_a - H_a{}^{\bar z}\bar\partial + \bar\partial H_a{}^{\bar z}\,)C^{\bar z} \qquad \text{for} \ \ a \,=\, z,\theta,\bar\theta \\
sC^{\bar z} &= -[\,C^{\bar z}\bar\partial C^{\bar z}\,] \\
s\Omega &= C^{\bar z}\bar\partial\Omega + (\bar\partial C^{\bar z})\Omega \ \ .
\end{aligned}
\tag{3.47}
$$

Altogether, the number of symmetry parameters and independent space-time fields coincide and the correspondence between them is given by

$$
\begin{array}{ccccccc}
C^z & C^\theta & C^{\bar\theta} & K & ; & C^{\bar z} \\
H_{\bar z}{}^z & H_{\bar\theta}{}^z & H_\theta{}^z & H_\theta{}^\theta / H_{\bar\theta}{}^{\bar\theta} & ; & H_\theta{}^{\bar z}, H_{\bar\theta}{}^{\bar z} \ .
\end{array}
\tag{3.48}
$$

Here, the superfields $H_\theta{}^{\bar z}$ and $H_{\bar\theta}{}^{\bar z}$ are constrained by chirality-type conditions which reduce the number of their components by a factor 1/2.

We note that the *holomorphic factorization* is manifestly realized for the s-variations (3.40)-(3.47) which have explicitly been verified to be nilpotent. The underlying *symmetry group* is the semi-direct product of superdiffeomorphisms and $U(1)$ transformations: this fact is best seen by rewriting the infinitesimal transformations of the ghost fields in terms of the ghost vector field $\Xi\cdot\partial$,

$$
\begin{aligned}
s\,(\Xi\cdot\partial) &= -\frac{1}{2}\,[\,\Xi\cdot\partial\,,\,\Xi\cdot\partial\,] \\
s\hat K &= -\,(\Xi\cdot\partial)\,\hat K \ \ .
\end{aligned}
\tag{3.49}
$$

Here, [,] denotes the graded Lie bracket and $\hat K = K - i_{\Xi\cdot\partial}V$ is a reparametrization of K involving the the $U(1)$ gauge field $V = e^a V_a$. More explicitly, we have

$$
\begin{aligned}
s\Xi^z &= -\Big[(\Xi\cdot\partial)\,\Xi^z \;-\; \Xi^\theta\,\Xi^{\bar\theta} \Big] \\
s\Xi^a &= -\,(\Xi\cdot\partial)\,\Xi^a \qquad\qquad\qquad \text{for } \; a=\bar z,\theta,\bar\theta \;\;,
\end{aligned}
\tag{3.50}
$$

where the quadratic term $\Xi^\theta\Xi^{\bar\theta}$ is due to the fact that the Ξ^a are the vector components with respect to the canonical tangent space basis (D_a) rather than the coordinate basis (∂_a).

Equations (3.44)(3.47) and some of the variations (3.45)-(3.46) involve only space-time derivatives and can be projected to component field expressions in a straightforward way [25, 26]. From the definitions

$$
\begin{aligned}
H_{\bar z}{}^z| &\equiv \mu_{\bar z}{}^z \quad , \quad H_{\bar z}{}^\theta| \equiv \alpha_{\bar z}{}^\theta \\
H_z{}^{\bar z}| &\equiv \bar\mu_z{}^{\bar z} \quad , \quad H_{\bar z}{}^{\bar\theta}| \equiv \bar\alpha_{\bar z}{}^{\bar\theta} \quad , \quad V_{\bar z}| \equiv \bar v_{\bar z}
\end{aligned}
\tag{3.51}
$$

and

$$
\begin{aligned}
C^z| &\equiv c^z \equiv \xi^z + \xi^{\bar z}\,\mu_{\bar z}{}^z \quad , \quad C^\theta| \equiv \epsilon^\theta \equiv \xi^\theta + \xi^{\bar z}\,\alpha_{\bar z}{}^\theta \\
C^{\bar z}| &\equiv \bar c^{\bar z} \equiv \xi^{\bar z} + \xi^z\,\bar\mu_z{}^{\bar z} \quad , \quad C^{\bar\theta}| \equiv \bar\epsilon^{\bar\theta} \equiv \xi^{\bar\theta} + \xi^{\bar z}\,\bar\alpha_{\bar z}{}^{\bar\theta} \\
K| &\equiv k \equiv \hat k + \xi^{\bar z}\,\bar v_{\bar z} \;\;,
\end{aligned}
\tag{3.52}
$$

we obtain the symmetry algebra of the ordinary Beltrami differentials $(\mu,\bar\mu)$, of their fermionic partners (the Beltraminos $\alpha,\bar\alpha$) and of the vector $\bar v$:

$$
\begin{aligned}
s\mu &= (\,\bar\partial - \mu\,\partial + \partial\mu\,)\,c - \bar\alpha\,\epsilon - \alpha\,\bar\epsilon \\
s\alpha &= (\,\bar\partial - \mu\,\partial + \frac{1}{2}\,\partial\mu + \bar v\,)\,\epsilon + c\,\partial\alpha + \frac{1}{2}\,\alpha\,\partial c + k\,\alpha \\
s\bar\alpha &= (\,\bar\partial - \mu\,\partial + \frac{1}{2}\,\partial\mu - \bar v\,)\,\bar\epsilon + c\,\partial\bar\alpha + \frac{1}{2}\,\bar\alpha\;\partial c - k\,\bar\alpha \\
s\bar v &= c\,\partial\bar v + \frac{1}{2}\,\alpha\,\partial\bar\epsilon - \frac{1}{2}\,\bar\epsilon\,\partial\alpha - \frac{1}{2}\,\bar\alpha\,\partial\epsilon + \frac{1}{2}\,\epsilon\,\partial\bar\alpha - (\,\bar\partial - \mu\,\partial\,)\,k \\
\\
sc &= c\,\partial c + \epsilon\,\bar\epsilon \\
s\epsilon &= c\,\partial\epsilon - \frac{1}{2}\,\epsilon\,\partial c + k\,\epsilon \\
s\bar\epsilon &= c\,\partial\bar\epsilon - \frac{1}{2}\,\bar\epsilon\,\partial c - k\,\bar\epsilon \\
sk &= c\,\partial k + \frac{1}{2}\,\epsilon\,\partial\bar\epsilon - \frac{1}{2}\,\bar\epsilon\,\partial\epsilon
\end{aligned}
\tag{3.53}
$$

and, for the $\bar z$-sector,

$$
\begin{aligned}
s\bar\mu &= (\,\partial - \bar\mu\,\bar\partial + \bar\partial\bar\mu\,)\,\bar c \\
s\bar c &= \bar c\,\bar\partial\bar c \;\;.
\end{aligned}
\tag{3.54}
$$

Thus, the holomorphic factorization remains manifestly realized at the component field level[3].

3.4 Scalar superfields

In (2,0) supersymmetry, ordinary scalar fields $X^i(z,\bar{z})$ generalize to complex superfields $\mathcal{X}^i$, $\bar{\mathcal{X}}^{\bar{\imath}} = (\mathcal{X}^i)^*$ satisfying the (anti-) chirality conditions

$$D_{\bar{\Theta}}\mathcal{X}^i \;=\; 0 \;=\; D_{\Theta}\bar{\mathcal{X}}^{\bar{\imath}} \quad . \tag{3.55}$$

The coupling of such fields to a superconformal class of metrics on the SRS $\mathbf{S\Sigma}$ is described by a sigma-model action [6, 7]:

$$\begin{aligned} S_{inv}[\mathcal{X},\bar{\mathcal{X}}] \;&=\; -\frac{i}{2}\int_{\mathbf{S\Sigma}} d^4Z\;[\,K_j(\mathcal{X},\bar{\mathcal{X}})\,\partial_{\bar{Z}}\mathcal{X}^j \;-\; \bar{K}_{\bar{\jmath}}(\mathcal{X},\bar{\mathcal{X}})\,\partial_{\bar{Z}}\bar{\mathcal{X}}^{\bar{\jmath}}\,] \\ &=\; -\frac{i}{2}\int_{\mathbf{S\Sigma}} d^4Z\;K_j(\mathcal{X},\bar{\mathcal{X}})\,\partial_{\bar{Z}}\mathcal{X}^j \;+\; \text{h.c.} \quad . \end{aligned} \tag{3.56}$$

Here, $d^4Z = dZ\,d\bar{Z}\,d\Theta\,d\bar{\Theta}$ and K_j denotes an arbitrary complex function (and $\bar{K}_{\bar{\jmath}} = (K_j)^*$ in the Minkowskian setting). The functional (3.56) is invariant under superconformal changes of coordinates for which the measure d^4Z transforms with $(D_\Theta\Theta')^{-1}\,(D_{\bar{\Theta}}\bar{\Theta}')^{-1}$, i.e. the Berezinian associated to the superconformal transformation (3.5)-(3.7).

We now rewrite the expression (3.56) in terms of the reference coordinates $(z,\bar{z},\theta,\bar{\theta})$ by means of Beltrami superfields. The passage from the small to the capital coordinates reads

$$\begin{pmatrix} \partial_Z \\ \partial_{\bar{Z}} \\ D_\Theta \\ D_{\bar{\Theta}} \end{pmatrix} \;=\; Q^{-1}\;M^{-1} \begin{pmatrix} \partial \\ \bar{\partial} \\ D \\ \bar{D} \end{pmatrix} \tag{3.57}$$

and the Berezinian of this change of variables is

$$\left| \frac{\partial(Z,\bar{Z},\Theta,\bar{\Theta})}{\partial(z,\bar{z},\theta,\bar{\theta})} \right| \;=\; \text{sdet}\,(M\,Q) \;=\; \Omega\,\text{sdet}\,M \quad . \tag{3.58}$$

The inverse of Q is easily determined:

$$Q^{-1} \;=\; \begin{pmatrix} \Lambda^{-1}\bar{\Lambda}^{-1} & 0 & -\Lambda^{-2}\bar{\Lambda}^{-1}\tau & -\Lambda^{-1}\bar{\Lambda}^{-2}\bar{\tau} \\ 0 & \Omega^{-1} & 0 & 0 \\ 0 & 0 & \Lambda^{-1} & 0 \\ 0 & 0 & 0 & \bar{\Lambda}^{-1} \end{pmatrix} \quad . \tag{3.59}$$

[3] In equations (3.53)(3.54), s is supposed to act from the left as usual in component field formalism and the graduation is given by the sum of the ghost-number and the Grassmann parity; the signs following from the superspace algebra have been modified so as to ensure nilpotency of the s-operation with these conventions.

In order to calculate sdet M and M^{-1}, we decompose M according to

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ h_\theta{}^z & h_\theta{}^{\bar z} & 1 & 0 \\ h_{\bar\theta}{}^z & h_{\bar\theta}{}^{\bar z} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & H_z{}^{\bar z} & 0 & 0 \\ H_{\bar z}{}^z & 1 & 0 & 0 \\ 0 & 0 & h_\theta{}^\theta & h_\theta{}^{\bar\theta} \\ 0 & 0 & h_{\bar\theta}{}^\theta & h_{\bar\theta}{}^{\bar\theta} \end{pmatrix} \begin{pmatrix} 1 & 0 & h_z{}^\theta & h_z{}^{\bar\theta} \\ 0 & 1 & h_{\bar z}{}^\theta & h_{\bar z}{}^{\bar\theta} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \quad (3.60)$$

The explicit expressions for the 'h' are

$$\begin{array}{rclcrcl} h_\theta{}^z &=& \Delta^{-1}(H_\theta{}^z - H_\theta{}^{\bar z} H_{\bar z}{}^z) &,& h_\theta{}^{\bar z} &=& \Delta^{-1}(H_\theta{}^{\bar z} - H_\theta{}^z H_z{}^{\bar z}) \\ h_{\bar\theta}{}^z &=& \Delta^{-1}(H_{\bar\theta}{}^z - H_{\bar\theta}{}^{\bar z} H_{\bar z}{}^z) &,& h_{\bar\theta}{}^{\bar z} &=& \Delta^{-1}(H_{\bar\theta}{}^{\bar z} - H_{\bar\theta}{}^z H_z{}^{\bar z}) \\ h_\theta{}^\theta &=& H_\theta{}^\theta - h_\theta{}^{\bar z} H_{\bar z}{}^\theta &,& h_\theta{}^{\bar\theta} &=& H_\theta{}^{\bar\theta} - h_\theta{}^{\bar z} H_{\bar z}{}^{\bar\theta} \\ h_{\bar\theta}{}^\theta &=& H_{\bar\theta}{}^\theta - h_{\bar\theta}{}^{\bar z} H_{\bar z}{}^\theta &,& h_{\bar\theta}{}^{\bar\theta} &=& H_{\bar\theta}{}^{\bar\theta} - h_{\bar\theta}{}^{\bar z} H_{\bar z}{}^{\bar\theta} \\ h_z{}^\theta &=& -\Delta^{-1} H_z{}^{\bar z} H_{\bar z}{}^\theta &,& h_z{}^{\bar\theta} &=& -\Delta^{-1} H_z{}^{\bar z} H_{\bar z}{}^{\bar\theta} \\ h_{\bar z}{}^\theta &=& \Delta^{-1} H_{\bar z}{}^\theta &,& h_{\bar z}{}^{\bar\theta} &=& \Delta^{-1} H_{\bar z}{}^{\bar\theta} \ , \end{array} \quad (3.61)$$

where $\Delta = 1 - H_z{}^{\bar z} H_{\bar z}{}^z$. It follows that sdet $M = \Delta/h$ with $h = h_\theta{}^\theta h_{\bar\theta}{}^{\bar\theta} - h_{\bar\theta}{}^\theta h_\theta{}^{\bar\theta}$ and that

$$M^{-1} = \begin{pmatrix} 1 & 0 & -h_z{}^\theta & -h_z{}^{\bar\theta} \\ 0 & 1 & -h_{\bar z}{}^\theta & -h_{\bar z}{}^{\bar\theta} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 1/\Delta & -H_z{}^{\bar z}/\Delta & 0 & 0 \\ -H_{\bar z}{}^z/\Delta & 1/\Delta & 0 & 0 \\ 0 & 0 & h_{\bar\theta}{}^{\bar\theta}/h & -h_\theta{}^{\bar\theta}/h \\ 0 & 0 & -h_{\bar\theta}{}^\theta/h & h_\theta{}^\theta/h \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -h_\theta{}^z & -h_\theta{}^{\bar z} & 1 & 0 \\ -h_{\bar\theta}{}^z & -h_{\bar\theta}{}^{\bar z} & 0 & 1 \end{pmatrix} .$$

From these results and equation (3.57), we can derive explicit expressions for $\partial_Z, \partial_{\bar Z}, D_\Theta, D_{\bar\Theta}$ which imply

$$\begin{array}{lcl} D_{\bar\Theta}\mathcal{X}^i = 0 & \Leftrightarrow & h_\theta{}^\theta(\bar D - h_{\bar\theta}{}^z\partial - h_{\bar\theta}{}^{\bar z}\bar\partial)\mathcal{X}^i \ = \ h_{\bar\theta}{}^\theta(D - h_\theta{}^z\partial - h_\theta{}^{\bar z}\bar\partial)\mathcal{X}^i \\ D_\Theta\bar{\mathcal{X}}^{\bar\imath} = 0 & \Leftrightarrow & h_{\bar\theta}{}^{\bar\theta}(D - h_\theta{}^z\partial - h_\theta{}^{\bar z}\bar\partial)\bar{\mathcal{X}}^{\bar\imath} \ = \ h_\theta{}^{\bar\theta}(\bar D - h_{\bar\theta}{}^z\partial - h_{\bar\theta}{}^{\bar z}\bar\partial)\bar{\mathcal{X}}^{\bar\imath} . \end{array} \quad (3.62)$$

Furthermore, by substituting $\partial_{\bar Z}$ into the action (3.56) and taking into account the last relation for $\mathcal{X}^i$, one obtains the *final result*

$$S_{inv}[\mathcal{X}, \bar{\mathcal{X}}] = -\frac{i}{2} \int_{\mathbf{S\Sigma}} d^4z \, K_j(\mathcal{X}, \bar{\mathcal{X}}) \, \bar\nabla \mathcal{X}^j \ + \ \text{h.c.} \ , \quad (3.63)$$

where $d^4z \ = \ dz \, d\bar z \, d\theta \, d\bar\theta$ and

$$\bar\nabla = \frac{1}{h}(\bar\partial - H_{\bar z}{}^z\partial) + \frac{1}{h^2} H_{\bar z}{}^\theta \left[h_\theta{}^{\bar\theta}(\bar D - h_{\bar\theta}{}^z\partial - h_{\bar\theta}{}^{\bar z}\bar\partial) - h_{\bar\theta}{}^{\bar\theta}(D - h_\theta{}^z\partial - h_\theta{}^{\bar z}\bar\partial) \right] . \quad (3.64)$$

3.5 Intermediate coordinates

If we disregard the complex conjugation relating z and $\bar{z}$, we can introduce the so-called intermediate or 'tilde' coordinates [26] by

$$(z,\bar{z},\theta,\bar{\theta}) \stackrel{M_1Q_1}{\longrightarrow} (\tilde{z},\tilde{\bar{z}},\tilde{\theta},\tilde{\bar{\theta}}) = (Z,\bar{z},\Theta,\bar{\Theta}) \stackrel{M_2Q_2}{\longrightarrow} (Z,\bar{Z},\Theta,\bar{\Theta}) \quad .$$

The matrix M_1Q_1 describing the passage from $(z,\bar{z},\theta,\bar{\theta})$ to $(\tilde{z},\tilde{\bar{z}},\tilde{\theta},\tilde{\bar{\theta}})$ is easy to invert: in analogy to eq.(3.57), we thus obtain the tilde derivatives

$$\begin{aligned}
\tilde{D} &= \frac{1}{\Lambda H}\left[H_{\bar{\theta}}{}^{\bar{\theta}}(D - H_{\theta}{}^{z}\partial) - H_{\theta}{}^{\bar{\theta}}(\bar{D} - H_{\bar{\theta}}{}^{z}\partial)\right] \\
\tilde{\bar{D}} &= \frac{1}{\bar{\Lambda} H}\left[H_{\theta}{}^{\theta}(\bar{D} - H_{\bar{\theta}}{}^{z}\partial) - H_{\bar{\theta}}{}^{\theta}(D - H_{\theta}{}^{z}\partial)\right] \\
\tilde{\partial} &= \frac{1}{\Lambda\bar{\Lambda}}\left[\partial - \tau\tilde{D} - \bar{\tau}\tilde{\bar{D}}\right] \\
\tilde{\bar{\partial}} &= (\bar{\partial} - H_{\bar{z}}{}^{z}\partial) - \Lambda H_{\bar{z}}{}^{\theta}\tilde{D} - \bar{\Lambda} H_{\bar{z}}{}^{\bar{\theta}}\tilde{\bar{D}} \quad ,
\end{aligned} \tag{3.65}$$

where $H = H_{\theta}{}^{\theta}H_{\bar{\theta}}{}^{\bar{\theta}} - H_{\bar{\theta}}{}^{\theta}H_{\theta}{}^{\bar{\theta}}$. For later reference, we note that $\mathrm{sdet}\,(M_1Q_1) = H^{-1}$.

For the passage from the tilde to the capital coordinates, we have

$$\begin{aligned}
D_{\Theta} &= \tilde{D} - k_{\theta}{}^{\bar{z}}\tilde{\bar{\partial}} \quad , & \partial_Z &= \tilde{\partial} - k_{z}{}^{\bar{z}}\tilde{\bar{\partial}} \\
D_{\bar{\Theta}} &= \tilde{\bar{D}} - k_{\bar{\theta}}{}^{\bar{z}}\tilde{\bar{\partial}} \quad , & \partial_{\bar{Z}} &= \Omega^{-1}\tilde{\bar{\partial}} \quad ,
\end{aligned}$$

where the explicit form of the 'k' in terms of the 'H' and $\Lambda,\bar{\Lambda}$ follows from the condition $MQ = (M_1Q_1)(M_2Q_2)$.

As a first application of the tilde coordinates, we prove that the solutions of the IFEQ's (3.30) for Λ and $\bar{\Lambda}$ are determined up to superconformal transformations of the capital coordinates, i.e. up to the rescalings (3.17). In fact, substitution of the expressions (3.28) for τ and $\bar{\tau}$ into the IFEQ's (3.30) shows that the homogenous equations associated to the IFEQ's can be rewritten as

$$\begin{aligned}
0 &= \tilde{D}\ln\Lambda = \tilde{\bar{\partial}}\ln\Lambda & &\Longrightarrow & &0 = D_{\Theta}\ln\Lambda = \partial_{\bar{Z}}\ln\Lambda \\
0 &= \tilde{\bar{D}}\ln\bar{\Lambda} = \tilde{\bar{\partial}}\ln\bar{\Lambda} & &\Longrightarrow & &0 = D_{\bar{\Theta}}\ln\bar{\Lambda} = \partial_{\bar{Z}}\ln\bar{\Lambda} \quad .
\end{aligned} \tag{3.66}$$

Henceforth, the solutions $\Lambda,\bar{\Lambda}$ of the IFEQ's are determined up to the rescalings

$$\begin{aligned}
\Lambda' &= \mathrm{e}^{f(Z,\Theta,\bar{\Theta})}\Lambda & &\text{with} \quad D_{\Theta}f = 0 \\
\bar{\Lambda}' &= \mathrm{e}^{g(Z,\Theta,\bar{\Theta})}\bar{\Lambda} & &\text{with} \quad D_{\bar{\Theta}}g = 0 \quad ,
\end{aligned}$$

which correspond precisely to the superconformal transformations (3.17).

Another application of the tilde coordinates consists of the determination of anomalies and effective actions and will be presented in section 3.8.

Since the z- and $\bar{z}$-sectors do not play a symmetric rôle in the (2,0)-theory, we can introduce a second set of intermediate coordinates which will be referred to as 'hat' coordinates:

$$(z,\bar{z},\theta,\bar{\theta}) \stackrel{\hat{M}_1\hat{Q}_1}{\longrightarrow} (\hat{z},\hat{\bar{z}},\hat{\theta},\hat{\bar{\theta}}) = (z,\bar{Z},\theta,\bar{\theta}) \stackrel{\hat{M}_2\hat{Q}_2}{\longrightarrow} (Z,\bar{Z},\Theta,\bar{\Theta}) \quad .$$

Using the hat derivatives

$$\begin{array}{rclcrcl} \hat{D} & = & D - H_\theta{}^{\bar{z}}\bar{\partial} & \quad , \quad & \hat{\partial} & = & \partial - H_z{}^{\bar{z}}\bar{\partial} \\ \hat{\bar{D}} & = & \bar{D} - H_{\bar{\theta}}{}^{\bar{z}}\bar{\partial} & \quad , \quad & \hat{\bar{\partial}} & = & \Omega^{-1}\bar{\partial} \quad , \end{array} \tag{3.67}$$

one proves that the ambiguity of the solutions of the IFEQ's for Ω coincides with superconformal rescalings.

By construction, the derivatives (3.67) satisfy the same algebra as the basic differential operators $(\partial,\bar{\partial},D,\bar{D})$, in particular,

$$\{\hat{D},\hat{\bar{D}}\} = \hat{\partial} \quad , \quad \hat{D}^2 = 0 = \hat{\bar{D}}^2 \quad , \quad [\hat{D},\hat{\partial}] = 0 = [\hat{\bar{D}},\hat{\partial}] \quad . \tag{3.68}$$

By virtue of these derivatives, the solution (3.33)(3.34) of the structure relations in the $\bar{z}$-sector can be rewritten in the compact form

$$H_z{}^{\bar{z}} = \hat{\bar{D}}H_\theta{}^{\bar{z}} + \hat{D}H_{\bar{\theta}}{}^{\bar{z}} \quad , \quad \hat{D}H_\theta{}^{\bar{z}} = 0 = \hat{\bar{D}}H_{\bar{\theta}}{}^{\bar{z}} \quad , \tag{3.69}$$

which equations will be further exploited in section 3.9.

3.6 Restriction of the geometry

In the study of the $N=1$ theory, it was noted that the choice $H_\theta{}^z = 0$ is invariant under superconformal transformations so that are no global obstructions for restricting the geometry by this condition. In fact, this choice greatly simplifies expressions involving Beltrami superfields and it might even be compulsory for the study of specific problems [32, 33]. As for the physical interpretation, the elimination of $H_\theta{}^z$ simply amounts to disregarding some pure gauge fields.

In the following, we introduce the $(2,0)$-analogon of the $N=1$ condition $H_\theta{}^z = 0$. In the present case, we have a greater freedom to impose conditions: this can be illustrated by the fact that a restriction of the form $DC^z = 0$ on the superdiffeomorphism parameter C^z does not imply $\partial C^z = 0$ (i.e. a restricted space-time dependence of C^z) as it does in the $N=1$ theory. The analogon of the $N=1$ restriction of the geometry is defined by the relations

$$H_\theta{}^z = 0 = H_{\bar{\theta}}{}^z \qquad \text{and} \qquad H_\theta{}^\theta \,/\, H_{\bar{\theta}}{}^{\bar{\theta}} = 1 \tag{3.70}$$

in the z-sector and

$$H_{\bar{\theta}}{}^{\bar{z}} = 0 \tag{3.71}$$

in the $\bar{z}$-sector. (The latter condition could also be replaced by $H_\theta{}^{\bar{z}} = 0$ since equations (3.26) following from the structure relations in the $\bar{z}$-sector are symmetric with respect to θ and $\bar{\theta}$.) Conditions (3.70) and (3.71) are compatible with the superconformal transformation laws (3.25).

In the remainder of the text, we will consider the geometry constrained by equations (3.70) and (3.71) which will be referred to as the *restricted geometry*. In this case, there is one unconstrained Beltrami superfield in the z-sector, namely $H_{\bar{z}}{}^z$, and one superfield in the $\bar{z}$-sector, namely $H_\theta{}^{\bar{z}}$, subject to the condition $(D - H_\theta{}^{\bar{z}}\bar{\partial})H_\theta{}^{\bar{z}} = 0$. The relations which hold for the other variables become

$$
\begin{array}{rcl}
D\Lambda & = & 0 \quad , \quad \tau = \bar{D}\Lambda \quad , \quad H_\theta{}^\theta = 1 \quad , \quad H_{\bar{\theta}}{}^\theta = 0 \quad , \quad H_{\bar{z}}{}^\theta = \bar{D}H_{\bar{z}}{}^z \\
\bar{D}\bar{\Lambda} & = & 0 \quad , \quad \bar{\tau} = D\bar{\Lambda} \quad , \quad H_{\bar{\theta}}{}^{\bar{\theta}} = 1 \quad , \quad H_\theta{}^{\bar{\theta}} = 0 \quad , \quad H_{\bar{z}}{}^{\bar{\theta}} = DH_{\bar{z}}{}^z \\
V_\theta & = & 0 \quad , \quad V_{\bar{\theta}} = 0 \quad , \quad V_{\bar{z}} = \dfrac{1}{2}\,[D,\bar{D}]H_{\bar{z}}{}^z \\
 & & \\
\bar{D}\Omega & = & 0 \quad , \quad H_z{}^{\bar{z}} = \bar{D}H_\theta{}^{\bar{z}} \quad , \quad (D - H_\theta{}^{\bar{z}}\bar{\partial})H_\theta{}^{\bar{z}} = 0 \quad ,
\end{array}
\tag{3.72}
$$

while the superconformal transformation laws now read

$$
\begin{array}{rcl}
\Lambda' & = & \mathrm{e}^w\,\Lambda \quad , \quad \bar{\Lambda}' = \mathrm{e}^{\bar{w}}\,\bar{\Lambda} \quad , \quad H_{\bar{z}'}{}^{z'} = \mathrm{e}^{-w-\bar{w}}\,(\bar{\partial}\bar{z}')^{-1}\,H_{\bar{z}}{}^z \\
 & & \\
\Omega' & = & (\bar{\partial}\bar{z}')^{-1}\,\Omega \quad , \quad H_{\theta'}{}^{\bar{z}'} = \mathrm{e}^w\,(\bar{\partial}\bar{z}')\,H_\theta{}^{\bar{z}} \quad .
\end{array}
$$

Furthermore, from (3.18) and (3.13), we get the local expressions

$$
\begin{array}{rcl}
\Lambda & = & D\Theta \qquad , \qquad \bar{\Lambda} = \bar{D}\bar{\Theta} \\
 & & \\
\Omega & = & \bar{\partial}\bar{Z} \qquad \text{(as before)} \quad .
\end{array}
$$

In order to be consistent, we have to require that the conditions (3.70) and (3.71) are invariant under the BRS transformations. This determines the symmetry parameters C^θ, $C^{\bar{\theta}}$, K in terms of C^z and eliminates some components of $C^{\bar{z}}$:

$$
\begin{array}{rcl}
C^\theta & = & \bar{D}C^z \quad , \quad C^{\bar{\theta}} = DC^z \quad , \quad K = \dfrac{1}{2}\,[D,\bar{D}]C^z \\
 & & \\
\bar{D}C^{\bar{z}} & = & 0 \quad .
\end{array}
\tag{3.73}
$$

The s-variations of the basic variables in the z-sector then take the form

$$
\begin{array}{rcl}
sH_{\bar{z}}{}^z & = & [\,\bar{\partial} - H_{\bar{z}}{}^z\partial - (\bar{D}H_{\bar{z}}{}^z)D - (DH_{\bar{z}}{}^z)\bar{D} + (\partial H_{\bar{z}}{}^z)\,]\,C^z \\
s\Lambda & = & [\,C^z\partial + (DC^z)\bar{D}\,]\,\Lambda \, + \, (D\bar{D}C^z)\,\Lambda \\
s\bar{\Lambda} & = & [\,C^z\partial + (\bar{D}C^z)D\,]\,\bar{\Lambda} \, + \, (\bar{D}DC^z)\,\bar{\Lambda} \\
sC^z & = & -\,[\,C^z\partial C^z + (\bar{D}C^z)(DC^z)\,] \quad ,
\end{array}
\tag{3.74}
$$

while those in the $\bar{z}$-sector are still given by equations (3.47).

Finite superdiffeomorphisms can be discussed along the lines of the $N = 1$ theory [26]. Here, we only note that the restriction (3.70)(3.71) on the geometry reduces the symmetry group sdiff $\mathbf{S\Sigma} \otimes U(1)$ to a subgroup thereof.

3.7 Component field expressions

In the restricted geometry (defined in the previous section), the basic variables of the z-sector are the superfields $H_{\bar{z}}{}^z$ and C^z which have the following θ-expansions:

$$\begin{aligned} H_{\bar{z}}{}^z &= \mu_{\bar{z}}{}^z + \theta\, \bar{\alpha}_{\bar{z}}{}^{\bar{\theta}} + \bar{\theta}\, \alpha_{\bar{z}}{}^{\theta} + \bar{\theta}\theta\, \bar{v}_{\bar{z}} \\ C^z &= c^z + \theta\, \bar{\epsilon}^{\bar{\theta}} + \bar{\theta}\, \epsilon^{\theta} + \bar{\theta}\theta\, k \quad . \end{aligned} \tag{3.75}$$

Here, the bosonic fields μ and $\bar{v}$ are the ordinary Beltrami coefficient and the $U(1)$ vector while α and $\bar{\alpha}$ represent their fermionic partners, the Beltraminos. These variables transform under general coordinate, local supersymmetry and local $U(1)$-transformations parametrized, respectively, by $c, \epsilon, \bar{\epsilon}$ and k.

The basic variables of the $\bar{z}$-sector are $H_{\theta}{}^{\bar{z}}$ and $C^{\bar{z}}$. To discuss their field content, we choose the *WZ-supergauge* in which the only non-vanishing component fields are

$$\bar{D} H_{\theta}{}^{\bar{z}} \,| = \bar{\mu}_z{}^{\bar{z}} \qquad \text{and} \qquad C^{\bar{z}} \,| = \bar{c}^{\bar{z}} \quad , \quad \bar{D} D C^{\bar{z}} \,| = \partial \bar{c}^{\bar{z}} \quad . \tag{3.76}$$

As expected for the (2,0)-supersymmetric theory, the $\bar{z}$-sector only involves the complex conjugate of μ and c.

In the remainder of this section, we present the component field results in the WZ-gauge. For the matter sector, we consider a single superfield $\mathcal{X}$ (and its complex conjugate $\bar{\mathcal{X}}$) and a flat target space metric ($K_j = \delta_{j\bar{\imath}}\, \bar{\mathcal{X}}^{\bar{\imath}}$). Henceforth, we only have one complex scalar and two spinor fields as component fields:

$$\begin{aligned} \mathcal{X} \,| &\equiv X \quad , \quad D\mathcal{X} \,| \equiv \lambda_{\theta} \\ \bar{\mathcal{X}} \,| &\equiv \bar{X} \quad , \quad \bar{D}\bar{\mathcal{X}} \,| \equiv \bar{\lambda}_{\bar{\theta}} \quad . \end{aligned} \tag{3.77}$$

For these fields, the invariant action (3.63) reduces to the following functional on the Riemann surface $\mathbf{\Sigma}$:

$$\begin{aligned} i\, S_{inv} = \int_{\mathbf{\Sigma}} d^2 z \, \Big\{ \frac{1}{1-\mu\bar{\mu}} \, \Big[& (\bar{\partial} - \mu\partial) X \, (\partial - \bar{\mu}\bar{\partial}) \bar{X} \\ & - \alpha\lambda(\partial - \bar{\mu}\bar{\partial})\bar{X} - \bar{\alpha}\bar{\lambda}(\partial - \bar{\mu}\bar{\partial})X - \bar{\mu}(\alpha\lambda)(\bar{\alpha}\bar{\lambda}) \Big] \\ & - \bar{\lambda}(\bar{\partial} - \mu\partial - \frac{1}{2}\partial\mu - \bar{v})\lambda \Big\} \quad . \end{aligned} \tag{3.78}$$

The s-variations of the matter superfields, $s\mathcal{X} = (\Xi \cdot \partial)\mathcal{X}$, $s\bar{\mathcal{X}} = (\Xi \cdot \partial)\bar{\mathcal{X}}$ can be projected to space-time in a straightforward manner: from the definitions $\Xi^z| \equiv \xi$, $\Xi^{\bar{z}}| \equiv \bar{\xi}$, $\Xi^\theta| \equiv \xi^\theta$, $\Xi^{\bar{\theta}}| \equiv \xi^{\bar{\theta}}$ and (3.75)-(3.77), it follows that

$$
\begin{aligned}
sX &= (\xi \cdot \partial)X + \xi^\theta \lambda \quad , \quad s\lambda = (\xi \cdot \partial)\lambda + \frac{1}{2}(\partial\xi + \mu\partial\bar{\xi})\lambda + \hat{k}\lambda + \xi^{\bar{\theta}}\mathcal{D}X \qquad (3.79)\\
s\bar{X} &= (\xi \cdot \partial)\bar{X} + \xi^{\bar{\theta}} \bar{\lambda} \quad , \quad s\bar{\lambda} = (\xi \cdot \partial)\bar{\lambda} + \frac{1}{2}(\partial\xi + \mu\partial\bar{\xi})\bar{\lambda} - \hat{k}\bar{\lambda} + \xi^{\theta}\mathcal{D}\bar{X} \ ,
\end{aligned}
$$

where we introduced the notation $\xi \cdot \partial \equiv \xi\partial + \bar{\xi}\bar{\partial}$, $\hat{k} \equiv k - \bar{\xi}\bar{v}$ and the supercovariant derivatives

$$
\mathcal{D}X = \frac{1}{1-\mu\bar{\mu}} \left[(\partial - \bar{\mu}\bar{\partial})X + \bar{\mu}\alpha\lambda \right] \quad , \quad \mathcal{D}\bar{X} = \frac{1}{1-\mu\bar{\mu}} \left[(\partial - \bar{\mu}\bar{\partial})\bar{X} + \bar{\mu}\bar{\alpha}\bar{\lambda} \right] \ . \qquad (3.80)
$$

3.8 Anomalies and effective actions

For the discussion of the chirally split form of the superdiffeomorphism anomaly and of its compensating action, we again consider the restricted geometry defined in section 3.6. We follow the procedure developed in reference [32] for the bosonic and $N = 1$ supersymmetric cases and we expect that the results can be extended to the *un*restricted geometry at the expense of technical complications as in the $N = 1$ case. We will mainly work on the superplane **SC**, but we will also comment on the generalization to generic compact SRS's. The results for the $\bar{z}$-sector are to be discussed in the next section.

The *holomorphically split form of the superdiffeomorphism anomaly* on the superplane is given in the z-sector by

$$
\begin{aligned}
\mathcal{A}^{(z)}[C^z; H_{\bar{z}}{}^z] &= \int_{\mathbf{SC}} d^4z \; C^z \, \partial[D, \bar{D}] \, H_{\bar{z}}{}^z \qquad (3.81)\\
&= \frac{1}{2} \int_{\mathbf{C}} d^2z \; \left\{ c\partial^3\mu + 2\epsilon\partial^2\bar{\alpha} + 2\bar{\epsilon}\partial^2\alpha + 4k\partial\bar{v} \right\} \ .
\end{aligned}
$$

It satisfies the Wess-Zumino (WZ) consistency condition $s\mathcal{A} = 0$. An expression which is well defined on a generic compact SRS is obtained by replacing the operator $\partial[D, \bar{D}]$ by the superconformally covariant operator

$$
\mathcal{L}_2 = \partial[D, \bar{D}] + \mathcal{R}\partial - (D\mathcal{R})\bar{D} - (\bar{D}\mathcal{R})D + (\partial\mathcal{R}) \qquad (3.82)
$$

depending on a superprojective connection $\mathcal{R}$ [34]; from $s\mathcal{R} = 0$, it follows that the so-obtained functional still satisfies the WZ consistency condition.

We note that our superspace expression for $\mathcal{A}$ was previously found in Polyakov's light-cone gauge [17] and that the corresponding component field expression coincides with the result found in reference [21] by differential geometric methods.

If written in terms of the tilde coordinates, the *Wess-Zumino-Polyakov (WZP) action* associated to the chirally split superdiffeomorphism anomaly on **SC** has the form of a free scalar field action for the integrating factor [32]. Thus, in the present case, it reads

$$S^{(z)}_{WZP}[H_{\bar{z}}{}^z] = \int_{\mathbf{SC}} d^4\tilde{z} \; \ln\bar{\Lambda}\,(\tilde{\bar{\partial}}\ln\Lambda) \quad , \tag{3.83}$$

where the variables $\ln\Lambda$ and $\ln\bar{\Lambda}$ represent (anti-) chiral superfields with respect to the tilde coordinates: $\tilde{D}\ln\Lambda = 0 = \tilde{\bar{D}}\ln\bar{\Lambda}$. By rewriting the action in terms of the coordinates $(z,\bar{z},\theta,\bar{\theta})$ and applying the s-operation, one reproduces the anomaly (3.81):

$$\begin{aligned} S^{(z)}_{WZP}[H_{\bar{z}}{}^z] &= -\int_{\mathbf{SC}} d^4z\; H_{\bar{z}}{}^z(\partial\ln\bar{\Lambda}) \\ sS^{(z)}_{WZP}[H_{\bar{z}}{}^z] &= -\mathcal{A}^{(z)}[C^z;H_{\bar{z}}{}^z] \quad . \end{aligned} \tag{3.84}$$

The response of the WZP-functional to an infinitesimal variation of the complex structure ($H_{\bar{z}}{}^z \to H_{\bar{z}}{}^z + \delta H_{\bar{z}}{}^z$) is given by the super Schwarzian derivative,

$$\frac{\delta S^{(z)}_{WZP}}{\delta H_{\bar{z}}{}^z} = \mathcal{S}(Z,\Theta;z,\theta) \quad , \tag{3.85}$$

the latter being defined by [30, 35, 34]

$$\mathcal{S}(Z,\Theta;z,\theta) = [D,\bar{D}]Q - (DQ)(\bar{D}Q) \qquad \text{with} \quad Q = \ln D\Theta + \ln\bar{D}\bar{\Theta} \quad . \tag{3.86}$$

The proof of this result proceeds along the lines of reference [32]: it makes use of the IFEQ's for $\Lambda = D\Theta$, $\bar{\Lambda} = \bar{D}\bar{\Theta}$ and of the fact that the functional (3.83) can be rewritten as

$$\begin{aligned} S^{(z)}_{WZP}[H_{\bar{z}}{}^z] &= \frac{1}{2}\int_{\mathbf{SC}} d^4\tilde{z}\; \left[\ln\bar{\Lambda}\;\tilde{\bar{\partial}}\ln\Lambda - \ln\Lambda\;\tilde{\bar{\partial}}\ln\bar{\Lambda}\right] \\ &= \frac{1}{2}\int_{\mathbf{SC}} d^4z\; \left[\ln\bar{\Lambda}\; D\bar{D}H_{\bar{z}}{}^z - \ln\Lambda\;\bar{D}DH_{\bar{z}}{}^z\right] \quad . \end{aligned} \tag{3.87}$$

Within the framework of (2,0) supergravity (i.e. the metric approach), the effective action $S^{(z)}_{WZP}$ represents a chiral gauge expression (see [32] and references therein): in this approach, it rather takes the form

$$S^{(z)}_{WZP} = -\int_{\mathbf{SC}} d^4z\; \frac{\partial\bar{\Theta}}{\bar{D}\bar{\Theta}}\;\bar{D}H_{\bar{z}}{}^z \quad , \tag{3.88}$$

which follows from (3.84) by substitution of $\bar{\Lambda} = \bar{D}\bar{\Theta}$.

We note that the extension of the WZP-action from **SC** to generic super Riemann surfaces has been discussed for the $N=0$ and $N=1$ cases in references [23, 36] and [37], respectively.

The *anomalous Ward identity* on the superplane reads

$$- \int_{\mathbf{SC}} d^4 z \; (sH_{\bar{z}}{}^z) \frac{\delta Z_c}{\delta H_{\bar{z}}{}^z} \; = \; k \, \mathcal{A}^{(z)}[C^z ; H_{\bar{z}}{}^z] \quad , \tag{3.89}$$

where Z_c denotes the vertex functional and k a constant. By substituting the explicit expression for $sH_{\bar{z}}{}^z$ and introducing the super stress tensor $\mathcal{T}_{\theta\bar{\theta}} = \delta Z_c \,/\, \delta H_{\bar{z}}{}^z$, the last equation takes the local form

$$\left[\bar{\partial} - H_{\bar{z}}{}^z \partial - (\bar{D} H_{\bar{z}}{}^z) D - (D H_{\bar{z}}{}^z) \bar{D} - (\partial H_{\bar{z}}{}^z) \right] \mathcal{T}_{\theta\bar{\theta}} \; = \; -k \, \partial [D, \bar{D}] H_{\bar{z}}{}^z \quad . \tag{3.90}$$

This relation has previously been derived and discussed in the light-cone gauge [17]. For $k \neq 0$, the redefinition $\mathcal{T} \to -k\mathcal{T}$ yields

$$\mathcal{L}_2 H_{\bar{z}}{}^z = \bar{\partial} \mathcal{T}_{\theta\bar{\theta}} \quad ,$$

where $\mathcal{L}_2$ represents the covariant operator (3.82) with $\mathcal{R} = \mathcal{T}$.

3.9 The $\bar{z}$-sector revisited

Since the hat derivatives $\hat{D}$ and $\hat{\bar{D}}$ are nilpotent, the constraint equations (3.69), i.e. $\hat{D} H_\theta{}^{\bar{z}} = 0 = \hat{\bar{D}} H_{\bar{\theta}}{}^{\bar{z}}$, can be solved in terms of superfields $H^{\bar{z}}$ and $\check{H}^{\bar{z}}$:

$$\begin{aligned} H_\theta{}^{\bar{z}} \;\; &= \;\; \hat{D} H^{\bar{z}} \; = \; (D - H_\theta{}^{\bar{z}} \bar{\partial}) H^{\bar{z}} \; = \; \sum_{n=0}^{\infty} (-\bar{\partial} H^{\bar{z}})^n \; D H^{\bar{z}} \\ H_{\bar{\theta}}{}^{\bar{z}} \;\; &= \;\; \hat{\bar{D}} \check{H}^{\bar{z}} \; = \; (\bar{D} - H_{\bar{\theta}}{}^{\bar{z}} \bar{\partial}) \check{H}^{\bar{z}} \; = \; \sum_{n=0}^{\infty} (-\bar{\partial} \check{H}^{\bar{z}})^n \; \bar{D} \check{H}^{\bar{z}} \quad . \end{aligned} \tag{3.91}$$

The last expression on the r.h.s. of these equations follows by iteration of the corresponding equation. The new variable $H^{\bar{z}}$ ($\check{H}^{\bar{z}}$) still allows for the addition of a superfield $G^{\bar{z}}$ ($\check{G}^{\bar{z}}$) satisfying $\hat{D} G^{\bar{z}} = 0$ ($\hat{\bar{D}} \check{G}^{\bar{z}} = 0$). The infinitesimal transformation laws of $H^{\bar{z}}$ and $\check{H}^{\bar{z}}$ read

$$\begin{aligned} sH^{\bar{z}} \;\; &= \;\; C^{\bar{z}} (1 + \bar{\partial} H^{\bar{z}}) + B^{\bar{z}} \quad , \quad sB^{\bar{z}} = -C^{\bar{z}} \bar{\partial} B^{\bar{z}} \qquad \text{with} \quad \hat{D} B^{\bar{z}} = 0 \\ s\check{H}^{\bar{z}} \;\; &= \;\; C^{\bar{z}} (1 + \bar{\partial} \check{H}^{\bar{z}}) + \check{B}^{\bar{z}} \quad , \quad s\check{B}^{\bar{z}} = -C^{\bar{z}} \bar{\partial} \check{B}^{\bar{z}} \qquad \text{with} \quad \hat{\bar{D}} \check{B}^{\bar{z}} = 0 \end{aligned} \tag{3.92}$$

and induce the transformation laws (3.47) of $H_\theta{}^{\bar{z}}$ and $H_{\bar{\theta}}{}^{\bar{z}}$.

We note that the introduction and transformation laws of $H^{\bar{z}}$ and $\check{H}^{\bar{z}}$ are very reminiscent of the prepotential V occuring in 4-dimensional supersymmetric Yang-Mills theories: in the abelian case, the latter transforms according to $sV = i(\Lambda - \bar{\Lambda})$ where Λ ($\bar{\Lambda}$) represents a chiral (anti-chiral) superfield.

For the restricted geometry, we have $\check{H}^{\bar{z}} = 0$ and, in the WZ-gauge, the non-vanishing component fields of $H^{\bar{z}}$ and $B^{\bar{z}}$ are

$$[D, \bar{D}] H^{\bar{z}} | \; = -2\bar{\mu} \qquad \text{and} \qquad B^{\bar{z}} | \; = -\bar{c} \quad , \quad [D, \bar{D}] B^{\bar{z}} | \; = -(\partial - 2\bar{\mu} \bar{\partial}) \bar{c} \; .$$

In this gauge, the *superdiffeomorphism anomaly in the $\bar{z}$-sector* takes the form

$$\mathcal{A}^{(\bar{z})}[C^{\bar{z}} ; H^{\bar{z}}] \; = \; \int_{\mathbf{SC}} d^4 z \, C^{\bar{z}} \bar{\partial}^3 H^{\bar{z}} \; = \; - \int_{\mathbf{C}} d^2 z \, \bar{c} \, \bar{\partial}^3 \bar{\mu} \quad . \tag{3.93}$$

3.10 Super Beltrami equations

Substitution of the expressions (3.13) into the definitions (3.15) yields the *super Beltrami equations*, e.g. the one involving the basic variable $H_{\bar{z}}{}^{z}$:

$$0 = (\bar{\partial} Z + \frac{1}{2}\bar{\Theta}\bar{\partial}\Theta + \frac{1}{2}\Theta\bar{\partial}\bar{\Theta}) - H_{\bar{z}}{}^{z}(\partial Z + \frac{1}{2}\bar{\Theta}\partial\Theta + \frac{1}{2}\Theta\partial\bar{\Theta}) \quad . \tag{3.94}$$

These equations can be used to define quasi-superconformal mappings [38, 30]; they occur in the supergravity approach [35] and have been studied from the mathematical point of view for the $N = 1$ case in reference [29].

Chapter 4

(2,2) Theory

4.1 Introduction

We now summarize the main results of the (2,2) theory. As expected, most expressions in the z-sector are the same as those of the (2,0) theory, while those in the $\bar{z}$-sector are simply obtained by complex conjugation. Therefore, our presentation closely follows the lines of chapter 3 and the new features are pointed out whenever they show up. The general framework for (2,2) SRS's and superconformal transformations is the one described in chapter 2.

4.2 Beltrami superfields

Starting from a reference complex structure given by local coordinates $(z, \theta, \bar{\theta}; \bar{z}, \theta^-, \bar{\theta}^-)$ on a (2,2) SRS, we pass over to a generic complex structure corresponding to local coordinates $(Z, \Theta, \bar{\Theta}; \bar{Z}, \Theta^-, \bar{\Theta}^-)$ by a smooth change of coordinates. The induced transformation law of the canonical 1-forms has the form

$$(e^Z, e^{\bar{Z}}, e^{\Theta}, e^{\bar{\Theta}}, e^{\Theta^-}, e^{\bar{\Theta}^-}) \;=\; (e^z, e^{\bar{z}}, e^{\theta}, e^{\bar{\theta}}, e^{\theta^-}, e^{\bar{\theta}^-}) \;\cdot M \cdot Q \quad , \qquad (4.1)$$

where the matrices M and Q contain the Beltrami superfields and integrating factors, respectively. More explicitly, MQ reads

$$\begin{pmatrix} 1 & H_z{}^{\bar{z}} & 0 & 0 & H_z{}^{\theta^-} & H_z{}^{\bar{\theta}^-} \\ H_{\bar{z}}{}^z & 1 & H_{\bar{z}}{}^{\theta} & H_{\bar{z}}{}^{\bar{\theta}} & 0 & 0 \\ H_\theta{}^z & H_\theta{}^{\bar{z}} & H_\theta{}^\theta & H_\theta{}^{\bar{\theta}} & H_\theta{}^{\theta^-} & H_\theta{}^{\bar{\theta}^-} \\ H_{\bar{\theta}}{}^z & H_{\bar{\theta}}{}^{\bar{z}} & H_{\bar{\theta}}{}^\theta & H_{\bar{\theta}}{}^{\bar{\theta}} & H_{\bar{\theta}}{}^{\theta^-} & H_{\bar{\theta}}{}^{\bar{\theta}^-} \\ H_{\theta^-}{}^z & H_{\theta^-}{}^{\bar{z}} & H_{\theta^-}{}^\theta & H_{\theta^-}{}^{\bar{\theta}} & H_{\theta^-}{}^{\theta^-} & H_{\theta^-}{}^{\bar{\theta}^-} \\ H_{\bar{\theta}^-}{}^z & H_{\bar{\theta}^-}{}^{\bar{z}} & H_{\bar{\theta}^-}{}^\theta & H_{\bar{\theta}^-}{}^{\bar{\theta}} & H_{\bar{\theta}^-}{}^{\theta^-} & H_{\bar{\theta}^-}{}^{\bar{\theta}^-} \end{pmatrix} \begin{pmatrix} \Lambda\bar{\Lambda} & 0 & \tau & \bar{\tau} & 0 & 0 \\ 0 & \Lambda^-\bar{\Lambda}^- & 0 & 0 & \tau^- & \bar{\tau}^- \\ 0 & 0 & \Lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\Lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 & \Lambda^- & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{\Lambda}^- \end{pmatrix} , \qquad (4.2)$$

where the indices $z, \theta, \bar{\theta}$ and $\bar{z}, \theta^-, \bar{\theta}^-$ are related by complex conjugation, e.g.

$$
\begin{array}{lllllllllll}
\Lambda^* & = & \Lambda^- & , & \tau^* & = \tau^- & , & (H_{\bar{z}}{}^z)^* & = & H_z{}^{\bar{z}} & , & (H_{\bar{\theta}}{}^\theta)^* & = & H_{\bar{\theta}^-}^{\theta^-} \\
\bar{\Lambda}^* & = & \bar{\Lambda}^- & , & \bar{\tau}^* & = \bar{\tau}^- & , & (H_\theta{}^z)^* & = & H_{\theta^-}^{\bar{z}} & , & & \dots &
\end{array}
$$

The 'H' are invariant under superconformal transformations of the capital coordinates while the integrating factors change under the latter according to

$$
\begin{array}{lllllll}
\Lambda' & = & \mathrm{e}^{-W}\, \Lambda & , & \bar{\Lambda}' & = & \mathrm{e}^{-\bar{W}}\, \bar{\Lambda} \\
\tau' & = & \mathrm{e}^{-W}\, [\, \tau \, - \, \Lambda \bar{\Lambda}\, (D_{\bar{\Theta}} W) \,] & , & \bar{\tau}' & = & \mathrm{e}^{-\bar{W}}\, [\, \bar{\tau} \, - \, \Lambda \bar{\Lambda}\, (D_{\Theta} \bar{W}) \,] \ ,
\end{array}
\qquad (4.3)
$$

where $\mathrm{e}^{-W} \equiv D_\Theta \Theta'$ and $\mathrm{e}^{-\bar{W}} \equiv D_{\bar{\Theta}} \bar{\Theta}'$. The transformation laws of $\Lambda^-, \bar{\Lambda}^-, \tau^-, \bar{\tau}^-$ are obtained by complex conjugation and involve $W^* = W^-, \bar{W}^* = \bar{W}^-$.

The $U(1)$ symmetry (with parameter K) of the (2,0) theory becomes a $U(1) \otimes U(1)$-symmetry parametrized by K and $K^- = K^*$ under which the fields transform according to

$$
\begin{array}{lllllll}
\Lambda' & = & \mathrm{e}^{K}\, \Lambda & , & \bar{\Lambda}' & = & \mathrm{e}^{-K}\, \bar{\Lambda} \\
(H_a{}^\theta)' & = & \mathrm{e}^{-K}\, H_a{}^\theta & , & (H_a{}^{\bar{\theta}})' & = & \mathrm{e}^{K}\, H_a{}^{\bar{\theta}} \qquad \text{for} \ \ a \neq z
\end{array}
\qquad (4.4)
$$

and the c.c. equations.

Due to the structure relations (2.7), the 'H' satisfy the following set of equations (and their c.c.):

$$
\begin{array}{rcl}
H_\theta{}^\theta H_{\bar{\theta}}{}^{\bar{\theta}} \, + \, H_{\bar{\theta}}{}^\theta H_\theta{}^{\bar{\theta}} & = & 1 \, - \, (\bar{D} - H_{\bar{\theta}}{}^z \partial) H_\theta{}^z \, - \, (D - H_\theta{}^z \partial) H_{\bar{\theta}}{}^z \\
H_{\theta^-}^{\theta} H_{\bar{\theta}^-}^{\bar{\theta}} \, + \, H_{\bar{\theta}^-}^{\theta} H_{\theta^-}^{\bar{\theta}} & = & H_{\bar{z}}{}^z \, - \, (\bar{D}_- - H_{\bar{\theta}^-}^{z} \partial) H_{\theta^-}^{z} \, - \, (D_- - H_{\theta^-}^{z} \partial) H_{\bar{\theta}^-}^{z} \qquad (4.5) \\
H_a{}^\theta H_a{}^{\bar{\theta}} & = & - \, (D_a - H_a{}^z \partial) H_a{}^z \qquad \text{for} \ \ a = \theta, \bar{\theta}, \theta^-, \bar{\theta}^- \\
H_{\bar{z}}{}^\theta H_a{}^{\bar{\theta}} \, + \, H_{\bar{z}}{}^{\bar{\theta}} H_a{}^\theta & = & (D_a - H_a{}^z \partial) H_{\bar{z}}{}^z \, - \, (\bar{\partial} - H_{\bar{z}}{}^z \partial) H_a{}^z \quad \text{for} \ \ a = \theta, \bar{\theta}, \theta^-, \bar{\theta}^- \\
H_a{}^\theta H_b{}^{\bar{\theta}} + H_b{}^\theta H_a{}^{\bar{\theta}} & = & - \, (D_a - H_a{}^z \partial) H_b{}^z \, - \, (D_b - H_b{}^z \partial) H_a{}^z \\
 & & \qquad \text{for} \ \ (a, b) = (\theta, \theta^-),\, (\theta, \bar{\theta}^-),\, (\bar{\theta}, \theta^-),\, (\bar{\theta}, \bar{\theta}^-) \ .
\end{array}
$$

By linearizing the variables ($H_\theta{}^\theta = 1 + h_\theta{}^\theta$, $H_{\bar{\theta}}{}^{\bar{\theta}} = 1 + h_{\bar{\theta}}{}^{\bar{\theta}}$ and $H_a{}^b = h_a{}^b$ otherwise), we find that the independent linearized fields are $h_\theta{}^z$, $h_{\bar{\theta}}{}^z$, $h_\theta{}^\theta - h_{\bar{\theta}}{}^{\bar{\theta}}$, $h_{\theta^-}^{z}$, $h_{\bar{\theta}^-}^{z}$ where the latter two satisfy (anti-) chirality conditions ($D_- h_{\theta^-}^{z} = 0 = \bar{D}_- h_{\bar{\theta}^-}^{z}$). Thus, there are 5 independent Beltrami superfields, $H_\theta{}^z$, $H_{\bar{\theta}}{}^z$, $H_{\theta^-}^{z}$, $H_{\bar{\theta}^-}^{z}$ and $H_\theta{}^\theta / H_{\bar{\theta}}{}^{\bar{\theta}}$, but $H_{\theta^-}^{z}$ and $H_{\bar{\theta}^-}^{z}$ satisfy chirality-type conditions which reduce the number of their independent component fields by a factor 1/2. In section 4.8, these constraints will be explicitly solved in a special case in terms of an unconstrained superfield H^z.

The factors τ, $\bar{\tau}$ are differential polynomials of the Beltrami coefficients and of the integrating factors $\Lambda, \bar{\Lambda}$:

$$
\begin{array}{lll}
\tau & = & (H_\theta{}^\theta H_{\bar{\theta}}{}^{\bar{\theta}} + H_{\bar{\theta}}{}^\theta H_\theta{}^{\bar{\theta}})^{-1} \left[(\bar{D} - H_{\bar{\theta}}{}^z \partial)(H_\theta{}^\theta \Lambda) + (D - H_\theta{}^z \partial)(H_{\bar{\theta}}{}^\theta \Lambda) \right] \\
\bar{\tau} & = & (H_\theta{}^\theta H_{\bar{\theta}}{}^{\bar{\theta}} + H_{\bar{\theta}}{}^\theta H_\theta{}^{\bar{\theta}})^{-1} \left[(D - H_\theta{}^z \partial)(H_{\bar{\theta}}{}^{\bar{\theta}} \bar{\Lambda}) + (\bar{D} - H_{\bar{\theta}}{}^z \partial)(H_\theta{}^{\bar{\theta}} \bar{\Lambda}) \right] \ .
\end{array}
\qquad (4.6)
$$

As for the factors $\Lambda, \bar{\Lambda}$ themselves, they satisfy the IFEQ's

$$
\begin{aligned}
0 &= (D_a - H_a{}^z \partial - \frac{1}{2} \partial H_a{}^z - V_a) \Lambda - H_a{}^{\bar\theta} \tau \\
0 &= (D_a - H_a{}^z \partial - \frac{1}{2} \partial H_a{}^z + V_a) \bar\Lambda - H_a{}^{\theta} \bar\tau \quad ,
\end{aligned}
\tag{4.7}
$$

where it is understood that $H_z{}^z = 1$ and $H_z{}^\theta = 0 = H_z{}^{\bar\theta}$. The c.c. variables $\Lambda^-, \bar\Lambda^-, \tau^-, \bar\tau^-$ satisfy the c.c. equations and the $U(1) \otimes U(1)$ connections V_a and V_a^- which appear in the previous set of equations are given by

$$
\begin{aligned}
V_z &= 0 \\
V_{\bar z} &= \frac{1}{H_\theta{}^\theta} \{ [D - H_\theta{}^z \partial + \frac{1}{2} (\partial H_\theta{}^z) + V_\theta] \, H_{\bar z}{}^\theta \; - \; [\bar\partial - H_{\bar z}{}^z \partial + \frac{1}{2} (\partial H_{\bar z}{}^z)] \, H_\theta{}^\theta \} \\
V_\theta &= - \frac{1}{H_\theta{}^\theta} \, [D - H_\theta{}^z \partial + \frac{1}{2} (\partial H_\theta{}^z)] \, H_\theta{}^\theta \\
V_{\bar\theta} &= \frac{1}{H_{\bar\theta}{}^{\bar\theta}} \, [\bar D - H_{\bar\theta}{}^z \partial + \frac{1}{2} (\partial H_{\bar\theta}{}^z)] \, H_{\bar\theta}{}^{\bar\theta} \\
V_a &= - \frac{1}{H_\theta{}^\theta} \{ [D_a - H_a{}^z \partial + \frac{1}{2} (\partial H_a{}^z)] \, H_\theta{}^\theta \; + \; [D - H_\theta{}^z \partial + \frac{1}{2} (\partial H_\theta{}^z) + V_\theta] \, H_a{}^\theta \} \\
& \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \text{for } \; a = \theta^-, \bar\theta^- \; .
\end{aligned}
\tag{4.8}
$$

We note that the last equations can also be written in the form

$$
\begin{aligned}
H_a{}^\theta V_a &= -[D_a - H_a{}^z \partial + \frac{1}{2} (\partial H_a{}^z)] \, H_a{}^\theta \qquad \text{for } \; a = \bar\theta, \theta^-, \bar\theta^- \\
H_a{}^{\bar\theta} V_a &= [D_a - H_a{}^z \partial + \frac{1}{2} (\partial H_a{}^z)] \, H_a{}^{\bar\theta} \qquad \text{for } \; a = \theta, \theta^-, \bar\theta^- \; .
\end{aligned}
\tag{4.9}
$$

4.3 Symmetry transformations

In order to obtain the transformation laws of the fields under infinitesimal superdiffeomorphisms and $U(1) \otimes U(1)$ transformations, we introduce the ghost vector field

$$
\Xi \cdot \partial \; \equiv \; \Xi^z \, \partial \; + \; \Xi^{\bar z} \, \bar\partial \; + \; \Xi^\theta \, D \; + \; \Xi^{\bar\theta} \, \bar D \; + \; \Xi^{\theta^-} \, D_- \; + \; \Xi^{\bar\theta^-} \, \bar D_- \quad ,
$$

(with $\Xi^a = \Xi^a(z, \theta, \bar\theta \, ; \bar z, \theta^-, \bar\theta^-)$) which generates an infinitesimal change of the coordinates $(z, \theta, \bar\theta \, ; \bar z, \theta^-, \bar\theta^-)$.

The $U(1) \otimes U(1)$ transformations again appear in a natural way in the transformation laws of the integrating factors and are parametrized by ghost superfields K and K^- . In terms of the reparametrized ghosts

$$
\left(C^z \, , \, C^{\bar z} \, , \, C^\theta \, , \, C^{\bar\theta} \, , \, C^{\theta^-} \, , \, C^{\bar\theta^-} \right) \; = \; \left(\Xi^z \, , \, \Xi^{\bar z} \, , \, \Xi^\theta \, , \, \Xi^{\bar\theta} \, , \, \Xi^{\theta^-} \, , \, \Xi^{\bar\theta^-} \right) \cdot M \quad , \tag{4.10}
$$

the BRS variations read

$$
\begin{array}{rcl}
s\Lambda & = & C^z\,\partial\Lambda \;+\; \dfrac{1}{2}\,(\partial C^z)\,\Lambda \;+\; C^{\bar\theta}\,\tau \;+\; K\,\Lambda \\
s\bar\Lambda & = & C^z\,\partial\bar\Lambda \;+\; \dfrac{1}{2}\,(\partial C^z)\,\bar\Lambda \;+\; C^{\theta}\,\bar\tau \;-\; K\,\bar\Lambda \\
s\tau & = & \partial\,(\,C^z\tau + C^{\theta}\Lambda\,) \\
s\bar\tau & = & \partial\,(\,C^z\bar\tau + C^{\bar\theta}\bar\Lambda\,) \quad ,
\end{array}
\qquad (4.11)
$$

$$
\begin{array}{rcl}
sH_a{}^z & = & (\,D_a - H_a{}^z\partial + \partial H_a{}^z\,)\,C^z - H_a{}^{\theta}C^{\bar\theta} - H_a{}^{\bar\theta}C^{\theta} \qquad (4.12) \\
sH_a{}^{\theta} & = & (\,D_a - H_a{}^z\partial + \dfrac{1}{2}\,\partial H_a{}^z + V_a\,)\,C^{\theta} + C^z\partial H_a{}^{\theta} - \dfrac{1}{2}\,H_a{}^{\theta}(\partial C^z) - H_a{}^{\theta}K \\
sH_a{}^{\bar\theta} & = & (\,D_a - H_a{}^z\partial + \dfrac{1}{2}\,\partial H_a{}^z - V_a\,)\,C^{\bar\theta} + C^z\partial H_a{}^{\bar\theta} - \dfrac{1}{2}\,H_a{}^{\bar\theta}(\partial C^z) + H_a{}^{\bar\theta}K \\
sV_a & = & C^z\partial V_a + \dfrac{1}{2}H_a{}^{\theta}\partial C^{\bar\theta} - \dfrac{1}{2}(\partial H_a{}^{\theta})C^{\bar\theta} - \dfrac{1}{2}H_a{}^{\bar\theta}\partial C^{\theta} + \dfrac{1}{2}(\partial H_a{}^{\bar\theta})C^{\theta} \\
 & & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +(D_a - H_a{}^z\partial)K
\end{array}
$$

$$
\begin{array}{rcl}
sC^z & = & -\,[\,C^z\partial C^z + C^{\theta}C^{\bar\theta}\,] \\
sC^{\theta} & = & -\,[\,C^z\partial C^{\theta} + \dfrac{1}{2}\,C^{\theta}(\partial C^z) - KC^{\theta}\,] \\
sC^{\bar\theta} & = & -\,[\,C^z\partial C^{\bar\theta} + \dfrac{1}{2}\,C^{\bar\theta}(\partial C^z) + KC^{\bar\theta}\,] \\
sK & = & -\,[\,C^z\partial K - \dfrac{1}{2}\,C^{\theta}(\partial C^{\bar\theta}) + \dfrac{1}{2}\,C^{\bar\theta}(\partial C^{\theta})\,] \quad .
\end{array}
\qquad (4.13)
$$

The variations of the c.c. fields are simply obtained by complex conjugation and henceforth the holomorphic factorization is manifestly realized for the chosen parametrization. Furthermore, the number of independent Beltrami fields and the number of symmetry parameters coincide. By projecting to space-time fields according to eqs.(3.51)(3.52), one obtains the transformation laws (3.53). The variations (4.12)(4.13) of $H_a{}^b$, V_a, C^a and K coincide with those found in the metric approach in reference [21].

4.4 Scalar superfields

We consider complex superfields $\mathcal{X}^i$ and $\bar{\mathcal{X}}^{\bar\imath} = (\mathcal{X}^i)^*$ satisfying the (twisted) chirality conditions [9]

$$
\begin{array}{rcl}
D_{\bar\Theta}\mathcal{X}^i & = & 0 \;=\; D_{\Theta^-}\mathcal{X}^i \\
D_{\Theta}\bar{\mathcal{X}}^{\bar\imath} & = & 0 \;=\; D_{\bar\Theta^-}\bar{\mathcal{X}}^{\bar\imath} \quad .
\end{array}
\qquad (4.14)
$$

Other multiplets have been introduced and discussed in references [9] and [12]. The sigma-model action describing the coupling of these fields to a superconformal class of metrics on the SRS $\mathbf{S\Sigma}$ is given by [39, 9]

$$S_{inv}[\mathcal{X}, \bar{\mathcal{X}}] = \int_{\mathbf{S\Sigma}} d^6Z \; K(\mathcal{X}, \bar{\mathcal{X}}) \quad , \tag{4.15}$$

where K is a real function of the fields $\mathcal{X}$, $\bar{\mathcal{X}}$ and $d^6Z = dZ\, d\bar{Z}\, d\Theta\, d\bar{\Theta}\, d\Theta^-\, d\bar{\Theta}^-$ is the superconformally invariant measure. For a flat target space metric, the functional (4.15) reduces to [3]

$$S_{inv}[\mathcal{X}, \bar{\mathcal{X}}] = \int_{\mathbf{S\Sigma}} d^6Z \; \mathcal{X}\bar{\mathcal{X}} \quad . \tag{4.16}$$

4.5 Restriction of the geometry

The restriction of the geometry is achieved by imposing the following conditions:

$$H_\theta{}^z = H_{\bar\theta}{}^z = H_{\bar\theta^-}^{\;z} = 0 \qquad \text{and} \qquad H_\theta{}^\theta / H_{\bar\theta}{}^{\bar\theta} = 1 \quad . \tag{4.17}$$

The addition of the c.c. equations goes without saying in this whole section. Equations (4.5) then imply that all Beltrami coefficients depend on $H_{\theta^-}^{\;z}$ by virtue of the relations

$$\begin{array}{rclcrclcrcl} H_{\bar z}{}^z &=& \bar D_- H_{\theta^-}^{\;z} &,& H_{\bar z}{}^\theta &=& \bar D H_{\bar z}{}^z &,& H_{\theta^-}^{\;\theta} &=& -\bar D H_{\theta^-}^{\;z} \\ &&&& H_{\bar z}{}^{\bar\theta} &=& D H_{\bar z}{}^z &,& H_{\theta^-}^{\;\bar\theta} &=& -D H_{\theta^-}^{\;z} \\ H_\theta{}^{\bar\theta} &=& H_{\bar\theta}{}^\theta = H_{\bar\theta^-}^{\;\theta} = H_{\bar\theta^-}^{\;\bar\theta} = 0 &,& && &&H_\theta{}^\theta &=& 1 = H_{\bar\theta}{}^{\bar\theta} \end{array} \tag{4.18}$$

and that $H_{\theta^-}^{\;z}$ itself satisfies the covariant chirality condition

$$(D_- - H_{\theta^-}^{\;z}\partial + DH_{\theta^-}^{\;z}\,\bar D)\, H_{\theta^-}^{\;z} = 0 \quad . \tag{4.19}$$

The relations satisfied by the other variables become

$$\begin{array}{rclcrclcrclcrcl} \tau &=& \bar D\Lambda &,& D\Lambda &=& 0 &,& \bar D_-\Lambda &=& 0 &,& D_-\Lambda &=& D\bar D(H_{\theta^-}^{\;z}\Lambda) \\ \bar\tau &=& D\bar\Lambda &,& \bar D\bar\Lambda &=& 0 &,& \bar D_-\bar\Lambda &=& 0 &,& D_-\bar\Lambda &=& \bar D D(H_{\theta^-}^{\;z}\bar\Lambda) \\ V_\theta &=& 0 &,& V_{\theta^-} &=& \frac{1}{2}[D,\bar D]H_{\theta^-}^{\;z} &,& V_{\bar z} &=& \bar D_- V_{\theta^-} && \\ V_{\bar\theta} &=& 0 &,& V_{\bar\theta^-} &=& 0 \quad . &&&&&&&& \end{array} \tag{4.20}$$

and eqs.(3.13)(3.18) yield the local expressions

$$\Lambda = D\Theta \quad , \quad \bar\Lambda = \bar D\bar\Theta \quad . \tag{4.21}$$

The s-invariance of conditions (4.17) implies that the symmetry parameters C^θ, $C^{\bar\theta}$ and K depend on C^z according to

$$C^{\bar\theta} = DC^z \quad , \quad C^\theta = \bar D C^z \quad , \quad K = \frac{1}{2}[D,\bar D]C^z \tag{4.22}$$

and that C^z itself satisfies the chirality condition

$$\bar{D}_- C^z \;=\; 0 \quad . \tag{4.23}$$

Thus, the s-variations of the basic variables read

$$\begin{aligned} sH_{\theta^-}{}^z &= \left[D_- - H_{\theta^-}{}^z \partial + (\bar{D} H_{\theta^-}{}^z) D + (D H_{\theta^-}{}^z) \bar{D} + (\partial H_{\theta^-}{}^z) \right] C^z \\ sC^z &= -\left[C^z \partial C^z + (DC^z)(\bar{D} C^z) \right] \quad . \end{aligned} \tag{4.24}$$

4.6 Intermediate coordinates

The intermediate coordinates which are relevant for us are those obtained by going over from z and $\bar{\theta}$ to capital coordinates without modifying the other coordinates:

$$(z, \theta, \bar{\theta} \,;\, \bar{z}, \theta^-, \bar{\theta}^-) \;\stackrel{M_1 Q_1}{\longrightarrow}\; (\tilde{z}, \tilde{\theta}, \tilde{\bar{\theta}} \,;\, \tilde{\bar{z}}, \tilde{\theta}^-, \tilde{\bar{\theta}}^-) \equiv (Z, \theta, \bar{\Theta} \,;\, \bar{z}, \theta^-, \bar{\theta}^-) \quad . \tag{4.25}$$

For the restricted geometry, we then get the explicit expression

$$\tilde{D}_- = D_- - H_{\theta^-}{}^z \partial + (D H_{\theta^-}{}^z) \bar{D} \tag{4.26}$$

and by construction we have $(\tilde{D}_-)^2 = 0$. Thus, the covariant chirality condition (4.19) for $H_{\theta^-}{}^z$ reads $\tilde{D}_- H_{\theta^-}{}^z = 0$ and may be solved by virtue of the nilpotency of the operator $\tilde{D}_-$ (see section 4.8).

4.7 Component field expressions

To write the action (4.15) in terms of the reference coordinates $(z, \theta, \bar{\theta} \,;\, \bar{z}, \theta^-, \bar{\theta}^-)$, we introduce the following superfields (as in the $(2,0)$ case):

$$\begin{aligned} h_a{}^z &= \Delta^{-1} (H_a{}^z - H_{\bar{z}}{}^z H_a{}^{\bar{z}}) \quad , & h_a{}^{\bar{z}} &= \Delta^{-1} (H_a{}^{\bar{z}} - H_z{}^{\bar{z}} H_a{}^z) \\ h_a{}^\theta &= H_a{}^\theta - h_a{}^{\bar{z}} H_{\bar{z}}{}^\theta \quad , & h_a{}^{\theta^-} &= H_a{}^{\theta^-} - h_a{}^z H_z{}^{\theta^-} \\ h_a{}^{\bar{\theta}} &= H_a{}^{\bar{\theta}} - h_a{}^{\bar{z}} H_{\bar{z}}{}^{\bar{\theta}} \quad , & h_a{}^{\bar{\theta}^-} &= H_a{}^{\bar{\theta}^-} - h_a{}^z H_z{}^{\bar{\theta}^-} \end{aligned} \tag{4.27}$$

for $a = \theta, \theta^-, \bar{\theta}, \bar{\theta}^-$. In the remainder of this section, we will consider the restricted geometry, for which the Berezinian takes the form

$$\left| \frac{\partial(Z, \Theta, \bar{\Theta} \,;\, \bar{Z}\, \Theta^-, \bar{\Theta}^-)}{\partial(z, \theta, \bar{\theta} \,;\, \bar{z}, \theta^-, \bar{\theta}^-)} \right| \;=\; \Delta / h \tag{4.28}$$

with $\Delta = 1 - H_{\bar{z}}{}^z H_z{}^{\bar{z}}$ and $h = h_\theta{}^\theta h_{\theta^-}{}^{\theta^-} - h_{\theta^-}{}^\theta h_\theta{}^{\theta^-}$. The chirality conditions for the matter superfields read $\bar{D}\mathcal{X} = 0$ and

$$h_\theta{}^\theta (D_- - h_{\theta^-}{}^{\bar{z}} \bar{\partial} - h_{\theta^-}{}^z \partial - h_{\theta^-}{}^{\bar{\theta}^-} \bar{D}_-) \mathcal{X} = h_{\theta^-}{}^\theta (D - h_\theta{}^{\bar{z}} \bar{\partial} - h_\theta{}^z \partial - h_\theta{}^{\bar{\theta}^-} \bar{D}_-) \mathcal{X} \tag{4.29}$$

and c.c. .

We now choose a *WZ-gauge* in which the basic superfields have the θ-expansions

$$
\begin{aligned}
H_{\theta^-}{}^{z} &= \bar\theta^-(\mu+\bar\theta\alpha+\theta\bar\alpha+\bar\theta\theta\bar v) \qquad , \quad C^z = c+\bar\theta\epsilon+\theta\bar\epsilon+\bar\theta\theta k \\
H_{\theta}{}^{\bar z} &= \bar\theta(\bar\mu+\bar\theta^-\alpha^-+\theta^-\bar\alpha^-+\bar\theta^-\theta^-\bar v^-) \ , \quad C^{\bar z} = \bar c+\bar\theta^-\epsilon^-+\theta^-\bar\epsilon^-+\bar\theta^-\theta^- k^- ,
\end{aligned}
\tag{4.30}
$$

whose form and physical interpretation is similar to the one of expressions (3.75) of the (2,0) theory. In fact, we have $H_{\theta^-}{}^{z} = \bar\theta^-\mathcal{H}_{\bar z}{}^{z}$ where $\mathcal{H}_{\bar z}{}^{z}$ denotes the basic Beltrami superfield of the (2,0) theory: a similar relationship holds in the WZ-gauge between the basic Beltrami superfields of the (1,1) and (1,0) theories [26].

The (twisted chiral) matter superfields $\mathcal{X}$ and $\bar{\mathcal{X}}$ contain one complex scalar, four spinors and one complex auxiliary fields as component fields [9, 12],

$$
\begin{aligned}
X = \mathcal{X}\,| \quad , \quad \lambda_\theta = D\mathcal{X}\,| \quad , \quad \bar\lambda^-_{\bar\theta^-} = \bar D_-\mathcal{X}\,| \quad , \quad F_{\theta\bar\theta^-} = D\bar D_-\mathcal{X}\,| \\
\bar X = \bar{\mathcal{X}}\,| \quad , \quad \lambda^-_{\theta^-} = D_-\bar{\mathcal{X}}\,| \quad , \quad \bar\lambda_{\bar\theta} = \bar D\bar{\mathcal{X}}\,| \quad , \quad \bar F_{\theta^-\bar\theta} = D_-\bar D\bar{\mathcal{X}}\,|
\end{aligned}
\tag{4.31}
$$

for which fields the action (4.16) reduces to the following functional on the Riemann surface $\mathbf{\Sigma}$:

$$
\begin{aligned}
S_{inv} = \int_{\mathbf{\Sigma}} d^2z \Big\{ \frac{1}{1-\mu\bar\mu} \, [\, & (\partial-\bar\mu\bar\partial)\bar X\,(\bar\partial-\mu\partial)X \ - \ \alpha\lambda(\partial-\bar\mu\bar\partial)\bar X \\
& - \alpha^-\lambda^-(\bar\partial-\mu\partial)X \ - \ \bar\alpha\bar\lambda(\partial-\bar\mu\bar\partial)X \ - \ \bar\alpha^-\bar\lambda^-(\bar\partial-\mu\partial)\bar X \\
& + (\alpha\lambda)(\alpha^-\lambda^- - \bar\mu\bar\alpha\bar\lambda) \ + \ (\bar\alpha^-\bar\lambda^-)(\bar\alpha\bar\lambda-\mu\alpha^-\lambda^-) \,] \\
& - \bar\lambda(\bar\partial-\mu\partial-\frac{1}{2}\partial\mu-\bar v)\lambda \ - \ \bar\lambda^-(\partial-\bar\mu\bar\partial-\frac{1}{2}\bar\partial\bar\mu-\bar v^-)\lambda^- \\
& - (1-\mu\bar\mu)\bar F F \Big\} \ .
\end{aligned}
\tag{4.32}
$$

In terms of $\xi^a = \Xi^a\,|$ and the short-hand notation

$$
\begin{aligned}
\xi \ &\equiv \ \xi^z \quad , \quad \hat k \equiv k - \bar\xi\bar v \quad , \quad \xi\cdot\partial \equiv \xi\partial+\bar\xi\bar\partial \\
\bar\xi \ &\equiv \ \xi^{\bar z} \quad , \quad \hat k^- \equiv k^- - \xi\bar v^- \quad ,
\end{aligned}
$$

the s-variations of the matter fields read

$$
\begin{aligned}
sX &= (\xi\cdot\partial)X + \xi^\theta\lambda + \xi^{\bar\theta^-}\bar\lambda^- \\
s\lambda &= [(\xi\cdot\partial) + \frac{1}{2}(\partial\xi+\mu\partial\bar\xi) + \hat k]\,\lambda \ + \ \xi^{\bar\theta}\mathcal{D}X \ - \ \xi^{\bar\theta^-}F \\
s\bar\lambda^- &= [(\xi\cdot\partial) + \frac{1}{2}(\bar\partial\bar\xi+\bar\mu\bar\partial\xi) - \hat k^-]\,\bar\lambda^- \ + \ \xi^{\theta^-}\bar{\mathcal{D}}X \ + \ \xi^\theta F \\
sF &= [(\xi\cdot\partial) + \frac{1}{2}(\partial\xi+\mu\partial\bar\xi) + \frac{1}{2}(\bar\partial\bar\xi+\bar\mu\bar\partial\xi) + \hat k - \hat k^-]\,F \\
& \qquad + \xi^{\bar\theta}\mathcal{D}\bar\lambda^- \ - \ \xi^{\theta^-}\bar{\mathcal{D}}\lambda \ ,
\end{aligned}
\tag{4.33}
$$

where we have introduced the supercovariant derivatives

$$\begin{aligned}
\mathcal{D}X &= \frac{1}{1-\mu\bar{\mu}}\left[(\partial-\bar{\mu}\bar{\partial})X+\bar{\mu}\alpha\lambda-\bar{\alpha}^{-}\bar{\lambda}^{-}\right] \\
\bar{\mathcal{D}}X &= \frac{1}{1-\mu\bar{\mu}}\left[(\bar{\partial}-\mu\partial)X+\mu\bar{\alpha}^{-}\bar{\lambda}^{-}-\alpha\lambda\right] \\
\mathcal{D}\bar{\lambda}^{-} &= \frac{1}{1-\mu\bar{\mu}}\left[(\partial-\bar{\mu}\bar{\partial}-\frac{1}{2}\bar{\partial}\bar{\mu}+\bar{v}^{-})\bar{\lambda}^{-}+\bar{\mu}\alpha F-\alpha^{-}\bar{\mathcal{D}}X\right] \\
\bar{\mathcal{D}}\lambda &= \frac{1}{1-\mu\bar{\mu}}\left[(\bar{\partial}-\mu\partial-\frac{1}{2}\partial\mu-\bar{v})\lambda-\mu\bar{\alpha}^{-}F-\bar{\alpha}\mathcal{D}X\right] \ .
\end{aligned} \tag{4.34}$$

A generic expression for the variations of the matter fields and for the supercovariant derivatives can be given in the supergravity framework where the component fields are defined by covariant projection [21]. We leave it as an exercise to check that the action (4.32) describing the superconformally invariant coupling of a twisted chiral multiplet to supergravity coincides with the usual component field expression [12] by virtue of the Beltrami parametrization of the space-time gauge fields (i.e. the zweibein, gravitino and $U(1)$ gauge field) - see [40, 15] for the $N = 1$ theory. Component field results for a chiral multiplet can be directly obtained from our results for the twisted chiral multiplet by application of the mirror map [12].

4.8 Anomaly

As pointed out in section 4.6, the constraint satisfied by $H_{\theta^-}{}^z$ in the restricted geometry, i.e. $\tilde{D}_- H_{\theta^-}{}^z = 0$, can be solved by virtue of the nilpotency of the operator $\tilde{D}_-$:

$$H_{\theta^-}{}^z = \tilde{D}_- H^z = \left[D_- - H_{\theta^-}{}^z\partial + (DH_{\theta^-}{}^z)\bar{D}\right] H^z \quad . \tag{4.35}$$

Here, the new variable H^z is determined up to a superfield G^z satisfying $\tilde{D}_- G^z = 0$ and it transforms according to

$$\begin{aligned}
sH^z &= C^z\,(1+\partial H^z)+(DC^z)(\bar{D}H^z)+B^z \qquad \text{with} \quad \tilde{D}_- B^z = 0 \\
sB^z &= -\left[C^z\partial B^z+(DC^z)(\bar{D}B^z)\right] \quad .
\end{aligned} \tag{4.36}$$

In the WZ-gauge, we have $H^z = \theta^- H_{\theta^-}{}^z$ with $H_{\theta^-}{}^z$ given by (4.30). In this case, the *holomorphically split form of the superdiffeomorphism anomaly* on the superplane reads

$$\begin{aligned}
\mathcal{A}[C^z;H^z] \ + \ \text{c.c.} &= \int_{\mathbf{SC}} d^6z\ C^z\,\partial[D,\bar{D}]\,H^z \ + \ \text{c.c.} \\
&= -\frac{1}{2}\int_{\mathbf{C}} d^2z\ \left\{c\partial^3\mu+2\epsilon\partial^2\bar{\alpha}+2\bar{\epsilon}\partial^2\alpha+4k\partial\bar{v}\right\} \ + \ \text{c.c.} \ \ .
\end{aligned} \tag{4.37}$$

It satisfies the consistency condition $s\mathcal{A} = 0$ and can be generalized to a generic compact SRS by replacing the operator $\partial[D, \bar{D}]$ by the superconformally covariant operator (3.82). The component field expression (4.37) coincides with the one found for the z-sector of the (2,0) theory, eq.(3.81), and with the one of references [41] and [21] where other arguments have been invoked.

At the linearized level, the transformation law (4.36) of H^z reads

$$\delta H^z = C^z + B^z \qquad \text{with} \quad \bar{D}_- C^z = 0 = D_- B^z \quad .$$

By solving the given constraints on C^z and B^z in terms of spinorial superfields L^θ and $L'^{\bar{\theta}}$, one finds

$$\delta H^z = \bar{D}_- L^\theta + D_- L'^{\bar{\theta}} \quad , \tag{4.38}$$

which result has the same form as the one found in the second of references [22], see eq.(3.19).

Chapter 5

Conclusion

In the course of the completion of our manuscript[1], the work [19] concerning the (2,0) theory appeared which also discusses the generalization of our previous $N = 1$ results [26, 32]. However, the author of reference [19] fails to take properly into account the U(1)-symmetry, connection and transformation laws which leads to incorrect results and conclusions. Furthermore, the super Beltrami coefficients (2.34) of [19] are not inert under superconformal transformations of the capital coordinates, eqs.(2.33), and therefore do not parametrize superconformal structures as they are supposed to. Finally, various aspects of the (2,0) theory that we treat here (e.g. superconformal models and component field expressions) are not addressed in reference [19].

In a supergravity approach [13] , some gauge choices are usually made when an explicit solution of the constraints is determined. Therefore, the question arises in which case the final solution represents a complete solution of the problem, i.e. a complete set of prepotentials (and compensators). Obviously, such a solution has been obtained if there are as many independent variables as there are independent symmetry parameters in the theory. If there is a smaller number of prepotentials, then it is clear that some basic symmetry parameters have been used to eliminate fields from the theory (a 'gauge choice' or 'restriction of the geometry' has been made). From these facts, we conclude that the solution of constraints discussed in references [16, 18, 19] and [22] is not complete. As for reference [21], it has not been investigated which ones are the independent variables.

Possible further developments or applications of our formalism include the derivation of operator product expansions and the proof of holomorphic factorization of partition functions along the lines of the work on the $N = 1$ theory [32, 27]. (The latter reference also involves the supersymmetric generalization of the Verlinde functional which occurs in conformal field theories and in the theory of W-algebras.) Another extension of the present study consists of the determination of $N = 2$ superconformally covariant differential operators and of their

[1] A preliminary version of the present paper has been part of the habilitation thesis of F.G. (Université de Chambéry, December 1994).

application to super W-algebras. This development will be reported on elsewhere [34].

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