Angular Momentum and Mutually Unbiased Bases

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Abstract

The Lie algebra of the group SU_2 is constructed from two deformed oscillator algebras for which the deformation parameter is a root of unity. This leads to an unusual quantization scheme, the $\{J^2, U_r\}$ scheme, an alternative to the familiar $\{J^2, J_z\}$ quantization scheme corresponding to common eigenvectors of the Casimir operator J^2 and the Cartan operator J_z. A connection is established between the eigenvectors of the complete set of commuting operators $\{J^2, U_r\}$ and mutually unbiased bases in spaces of constant angular momentum.

Key words: angular momentum; deformations; harmonic oscillator; Lie algebra; polar decomposition; MUBs.

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1 Introduction

In recent years, the notion of deformed oscillator algebra and its extension to deformed Lie algebra, or Hopf algebra in mathematical parlance,[1−5] proved to be useful in various fields of theoretical physics. For instance, one- and two-parameter deformations of oscillator algebras and Lie algebras were successfully applied to statistical mechanics[6−13] and to nuclear, atomic and molecular physics.[14−18] In the case where the deformation parameter is a root of unity, let us also mention the importance of deformed oscillator algebras for the definition of k-fermions, which are objects interpolating between fermions and bosons,[19] and the study of fractional supersymmetry.[20]

The aim of this note is two-fold. First, we show how a deformation of two truncated harmonic oscillators leads to a polar decomposition of the Lie algebra of SU_2. Such a decomposition is especially appropriate for developing the representation theory and the Wigner–Racah algebra of SU_2 in a non-standard basis adapted to cyclical symmetry.[21] Second, we establish a contact between the corresponding bases for spaces of constant angular momentum and the so-called mutually unbiased bases (MUBs) in a finite-dimensional Hilbert space. The latter bases[22−44] play a central role in quantum information theory. In particular, the use of quantum-mechanical states belonging to MUBs is of paramount importance in quantum cryptography (securing quantum key exchange) and quantum state tomography (deciphering a quantum state).

2 Angular Momentum Theory in a Nonstandard Basis

2.1 The Lie algebra of SU_2 from two oscillator algebras

Let $\mathcal{F}(1)$ and $\mathcal{F}(2)$ be two finite-dimensional Hilbert spaces of dimension k with $k \in \mathbf{N} \setminus \{0, 1\}$. We use $(\mid)$ to denote the inner product on $\mathcal{F}(i)$ and, for each space $\mathcal{F}(i)$ with $i = 1, 2$, we choose an orthonormal basis $\{|n_i) : n_i = 0, 1, \cdots, k-1\}$. Let (a_{i-}, a_{i+}, N_i) be a triplet of linear operators on $\mathcal{F}(i)$ defined by

$$a_{i\pm}|n_i) = \left(\left[n_i + s \pm \frac{1}{2}\right]_q\right)^{\alpha_{i\pm}} |n_i \pm 1)$$

$$a_{i+}|k-1) = 0, \quad a_{i-}|0) = 0, \quad N_i|n_i) = n_i|n_i)$$

where

$$s = \frac{1}{2}, \quad \alpha_{i\pm} = \frac{1 \pm (-1)^i}{2}, \quad q = \exp\left(\frac{2\pi \mathrm{i}}{k}\right), \quad [x]_q = \frac{1 - q^x}{1 - q}, \quad x \in \mathbf{R}$$

with $i = 1, 2$. It can be shown that the operators a_{i-}, a_{i+} and N_i satisfy the following relations

$$a_{i-}a_{i+} - q a_{i+}a_{i-} = 1, \quad (a_{i\pm})^k = 0, \quad [N_i, a_{i\pm}] = \pm a_{i\pm}, \quad N_i^\dagger = N_i \tag{1}$$

where we use the notation $A^\dagger$ for the adjoint of A and $[A, B]$ for the commutator of the operators A and B. The two algebras defined by Eq. (1) with $i = 1, 2$ are two commuting oscillator algebras with q being a root of unity; this is reminiscent of the two oscillator algebras used for the introduction of k-fermions.[19,20]

We now consider the space $\mathcal{F}_k = \mathcal{F}(1) \otimes \mathcal{F}(2)$ of dimension k^2. An orthonormal basis for $\mathcal{F}_k$ is provided by the vectors

$$|n_1, n_2) = |n_1) \otimes |n_2), \quad n_i = 0, 1, \cdots, k-1, \quad i = 1, 2$$

The key of our derivation of a nonstandard basis of SU_2 consists in defining the two linear operators

$$H = \sqrt{N_1 \, (N_2 + 1)}, \quad U_r = s_{1+} s_{2-}$$

where

$$s_{i\pm} = a_{i\pm} + \mathrm{e}^{\frac{1}{2}\mathrm{i}\phi_r} \frac{1}{[k-1]_q!} (a_{i\mp})^{k-1}$$

for $i = 1, 2$. In the operator $s_{i\pm}$, the phase ϕ_r is an arbitrary real parameter taken in the form

$$\phi_r = \pi(k-1)r, \quad r \in \mathbf{R}$$

and $[n]_q!$ stands for the q-deformed factorial defined by

$$\forall n \in \mathbf{N}^* : [n]_q! = [1]_q \, [2]_q \cdots [n]_q \, , \quad [0]_q! = 1$$

It is immediate to show that the action of H and U_r on $\mathcal{F}_k$ is given by

$$H|n_1, n_2) = \sqrt{n_1(n_2+1)} |n_1, n_2), \quad n_i = 0, 1, 2, \cdots, k-1, \quad i = 1, 2$$

and

$$U_r|n_1, n_2) = |n_1 + 1, n_2 - 1), \quad n_1 \neq k-1, \quad n_2 \neq 0$$

$$U_r|k-1, n_2) = \mathrm{e}^{\frac{1}{2}\mathrm{i}\phi_r}|0, n_2 - 1), \quad n_2 \neq 0$$

$$U_r|n_1, 0) = \mathrm{e}^{\frac{1}{2}\mathrm{i}\phi_r}|n_1 + 1, k-1), \quad n_1 \neq k-1$$

$$U_r|k-1, 0) = \mathrm{e}^{\mathrm{i}\phi_r}|0, k-1)$$

The operators H and U_r satisfy interesting properties. The operator H is Hermitean and the operator U_r is unitary. Furthermore, the action of U_r on the space $\mathcal{F}_k$ is cyclic in the sense that

$$(U_r)^k = \mathrm{e}^{\mathrm{i}\phi_r} I$$

where I is the identity operator.

From the Schwinger work on angular momentum,[45] we introduce

$$J = \frac{1}{2}\left(n_1 + n_2\right), \quad M = \frac{1}{2}\left(n_1 - n_2\right)$$

We shall use the notation

$$|JM\rangle \equiv |J+M, J-M) = |n_1, n_2)$$

For a fixed value of J, the label M can take $2J+1$ values $M = -J, -J+1, \cdots, J$. For fixed k, the maximum value of J is $J = J_{\max} = k-1$ and the following value of J

$$J = j = \frac{1}{2}(k-1)$$

is admissible. For a given value of $k \in \mathbf{N} \setminus \{0,1\}$, the $2j+1 = k$ vectors $|jm\rangle$ belong to the vector space $\mathcal{F}_k$. Let $\varepsilon(j)$ be the subspace of $\mathcal{F}_k$, of dimension $\dim \varepsilon(j) = k$, spanned by the k vectors $|jm\rangle$ with $m = -j, -j+1, \cdots, j$. We can thus associate the space $\varepsilon(j)$ for $j = \frac{1}{2}, 1, \frac{3}{2}, \cdots$ to the values $k = 2, 3, 4, \cdots$, respectively. The subspace $\varepsilon(j)$ of $\mathcal{F}_k$ is stable under H and U_r. Indeed, the action of the operators H and U_r on the space $\varepsilon(j)$ can be described by

$$H|jm\rangle = \sqrt{(j+m)(j-m+1)}|jm\rangle$$

and

$$U_r|jm\rangle = \left[1 - \delta(m,j)\right]|jm+1\rangle + \delta(m,j)\mathrm{e}^{\mathrm{i}\phi_r}|j-j\rangle$$

We can check that the operator H is Hermitean and the operator U_r is unitary on the space $\varepsilon(j)$. Furthermore, we have $(U_r)^{2j+1} = \mathrm{e}^{\mathrm{i}\phi_r} I$ which reflects the cyclical character of U_r on $\varepsilon(j)$.

We are now in a position to give a realization of the Lie algebra of the group SU_2 in terms of U_r, N_1 and N_2. Let us define the three operators

$$J_+ = HU_r, \quad J_- = U_r^\dagger H, \quad J_z = \frac{1}{2}\left(N_1 - N_2\right)$$

It is straightforward to check that

$$J_\pm|jm\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|jm \pm 1\rangle, \quad J_z|jm\rangle = m|jm\rangle$$

Consequently, we get the commutation relations

$$[J_z, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_z$$

which correspond to the Lie algebra of SU_2.

2.2 An nonstandard basis for the group SU_2

The decomposition of the shift operators J_+ and J_- in terms of H and U_r coincides with the polar decomposition worked out in Refs. 46 and 47 in a completely different way. This is easily seen by taking the matrix elements of U_r and H in the $\{J^2, J_z\}$ quantization scheme and by comparing these elements to the ones of the operators Υ and J_T in Ref. 46. We are thus left with $H = J_T$ and, by identifying the arbitrary phase φ of Ref. 46 with ϕ_r, we obtain $U_r = \Upsilon$ so that $J_+ = J_T \Upsilon$ and $J_- = \Upsilon^\dagger J_T$.

It is immediate to check that the Casimir operator J^2 of the Lie algebra su_2 can be rewritten as

$$J^2 = \frac{1}{4}(N_1 + N_2)(N_1 + N_2 + 2)$$

in terms of N_1 and N_2. It is a simple matter of calculation to prove that J^2 commutes with U_r for any value of r. Therefore, for r fixed, the commuting set $\{J^2, U_r\}$ provides us with an alternative to the familiar commuting set $\{J^2, J_z\}$ of angular momentum theory.

The eigenvalues and the common eigenvectors of the complete set of commuting operators $\{J^2, U_r\}$ can be easily found. This leads to the following result.

Result 1. The spectra of the operators U_r and J^2 are given by

$$U_r |j n_\alpha; r\rangle = q^{-\alpha} |j n_\alpha; r\rangle, \quad J^2 |j n_\alpha; r\rangle = j(j+1) |j n_\alpha; r\rangle$$

where

$$|j n_\alpha; r\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} q^{\alpha m} |jm\rangle, \quad q = \exp\left(\mathrm{i}\frac{2\pi}{2j+1}\right) \tag{2}$$

with the range of values

$$\alpha = -jr + n_\alpha, \quad n_\alpha = 0, 1, \cdots, 2j$$

where $2j \in \mathbf{N}^*$ and $r \in \mathbf{R}$.

Each vector $|j n_\alpha; r\rangle$ can be considered as a discrete Fourier transform[47] in the finite-dimensional Hilbert space $\varepsilon(j)$. As a matter of fact, the inter-basis expansion coefficients

$$\langle jm | j n_\alpha; r\rangle = \frac{1}{\sqrt{2j+1}} \exp\left[\mathrm{i}\frac{2\pi}{2j+1}(-jr + n_\alpha)m\right]$$

(with $m = -j, -j+1, \cdots, j$ and $n_\alpha = 0, 1, \cdots, 2j$) in Eq. (2) define a unitary transformation, in $\varepsilon(j)$ (with $j = \frac{1}{2}, 1, \frac{3}{2}, \cdots$), that allows to pass from the well-known orthonormal standard spherical basis

$$s(j) = \{|jm\rangle : m = -j, -j+1, \cdots, j\}$$

to the orthonormal non-standard basis

$$b_r(j) = \{|j n_\alpha; r\rangle : n_\alpha = 0, 1, \cdots, 2j\}$$

For a given value of r, the basis $b_r(j)$ is an alternative to the spherical basis $s(j)$ of the space $\varepsilon(j)$. Two bases $b_r(j)$ and $b_s(j)$ with $r \neq s$ are thus two equally admissible orthonormal bases for $\varepsilon(j)$. The state vectors of the bases $b_r(j)$ and $b_s(j)$ are common eigenstates of $\{J^2, U_r\}$ and $\{J^2, U_s\}$, respectively. The overlap between the bases $b_r(j)$ and $b_s(j)$ is controlled by

$$\langle jn_\alpha; r|jn_\beta; s\rangle = \frac{1}{2j+1}\frac{\sin(\alpha-\beta)\pi}{\sin(\alpha-\beta)\frac{\pi}{2j+1}}$$

with $\alpha = -jr + n_\alpha$ and $\beta = -js + n_\beta$ where $n_\alpha, n_\beta = 0, 1, \cdots, 2j$.

3 Connection with Mutually Unbiased Bases

We are now ready for establishing contact with MUBs. Let $\varepsilon(d)$ be a Hilbert space of dimension d endowed with the inner product $\langle\ |\ \rangle$. Two orthonormal bases $A = \{|A\alpha\rangle : \alpha = 0, 1, \cdots, d-1\}$ and $B = \{|B\beta\rangle : \beta = 0, 1, \cdots, d-1\}$ are said to be mutually unbiased if and only if $|\langle A\alpha|B\beta\rangle| = \frac{1}{\sqrt{d}}$ for all $\alpha \in \{0, 1, \cdots, d-1\}$ and all $\beta \in \{0, 1, \cdots, d-1\}$. For an arbitrary value of d, the number of MUBs cannot be greater than $d+1$.[22–25]

We note in passing that the latter result can be justified from group theory. The d orthonormal vectors of a basis for $\varepsilon(d)$ can be considered as a basis for a fundamental representation of dimension d of the group SU_d. This group is of dimension d^2-1 and of rank (i.e., the number of Cartan generators) $d-1$. Therefore, the maximal number of independent sets of $d-1$ commuting operators it is possible to construct from the d^2-1 generators of SU_d is $\frac{d^2-1}{d-1} = d+1$. This is precisely the maximum number of MUBs for the space $\varepsilon(d)$. Indeed, the limit $d+1$ is reached if d is a prime number[23] or a power of a prime number.[24–30]

It is also interesting to note that a connection exists between MUBs and various geometries (e.g., see Refs. 31, 36, 38, 40 and 43). In particular, according to the SPR conjecture,[31] for d fixed with d not equal to a power of a prime number, the problem of the existence of a complete set of $d+1$ MUBs would be equivalent to the one of the existence of projective planes of order d.

We derive below some preliminary results of interest for an investigation of a relation between the $\{J^2, U_r\}$ scheme and MUBs. To begin with, from Eq. (2), we have the following result.

Result 2. The overlap between the bases $s(j)$ and $b_r(j)$ satisfies

$$|\langle jm|jn_\alpha; r\rangle|^2 = \frac{1}{\dim \varepsilon(j)}$$

so that $s(j)$ and $b_r(j)$ are two MUBs for the space $\varepsilon(j)$.

As an illustration, we consider the space $\varepsilon(\frac{1}{2})$ of dimension 2. Equation (2) yields

$$|\frac{1}{2}0;0\rangle = \frac{1}{\sqrt{2}}\left(|\frac{1}{2} - \frac{1}{2}\rangle + |\frac{1}{2}\frac{1}{2}\rangle\right), \quad |\frac{1}{2}1;0\rangle = \frac{\mathrm{i}}{\sqrt{2}}\left(-|\frac{1}{2} - \frac{1}{2}\rangle + |\frac{1}{2}\frac{1}{2}\rangle\right)$$

for $r = 0$ and

$$|\frac{1}{2}0;1\rangle = \frac{1}{\sqrt{2}}\left(\rho|\frac{1}{2}-\frac{1}{2}\rangle + \rho^{-1}|\frac{1}{2}\frac{1}{2}\rangle\right), \quad |\frac{1}{2}1;1\rangle = \frac{1}{\sqrt{2}}\left(\rho^{-1}|\frac{1}{2}-\frac{1}{2}\rangle + \rho|\frac{1}{2}\frac{1}{2}\rangle\right)$$

for $r = 1$ with $\rho = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$. It is evident that the three bases $s(\frac{1}{2})$, $b_0(\frac{1}{2})$ and $b_1(\frac{1}{2})$ constitute a complete set of MUBs for $\varepsilon(\frac{1}{2})$.

The situation is not so simple for $2j \in \mathbf{N} \setminus \{0, 1\}$. For fixed j, the eigenfunctions of the operators U_r and U_s, with $r \neq s$, are not necessarily independent. We give in what follows some results that can be useful for $2j \neq 1$.

Result 3. By assuming

$$s = r + \frac{n_\beta - n_\alpha}{j} + \frac{2j+1}{j}k_{\alpha\beta}, \quad k_{\alpha\beta} \in \mathbf{Z} \tag{3}$$

we get

$$|jn_\beta; s\rangle = (-1)^{2jk_{\alpha\beta}}|jn_\alpha; r\rangle$$

and the corresponding bases $b_r(j)$ and $b_s(j)$ are not MUBs.

Result 4. The commutator of U_s and U_r on $\varepsilon(j)$ assumes the form

$$[U_s, U_r] = \left(\mathrm{e}^{\mathrm{i}\phi_s} - \mathrm{e}^{\mathrm{i}\phi_r}\right)[|j,-j\rangle\langle j, j-1| - |j,-j+1\rangle\langle j,j|]$$

Therefore, a necessary and sufficient condition that the operators U_s and U_r commute is

$$s = r + \frac{x}{j}, \quad x \in \mathbf{Z} \tag{4}$$

We note that Eq. (3) implies Eq. (4).

Result 5. On the space $\varepsilon(j)$, let Z be the familiar phase operator defined by

$$\forall m \in \{-j, -j+1, \cdots, j\} : Z|jm\rangle = q^{-m}|jm\rangle$$

and, for fixed r, let V_{ra} be the $2j+1$ unitary operators given by

$$V_{ra} = U_r Z^a = q^a Z^a U_r, \quad a = 0, 1, \cdots, 2j$$

(cf. the Weyl commutation relation rule). The Hilbert-Schmidt inner product of the operators V_{sb} and V_{ra} is

$$\mathrm{tr}\left(V_{sb}^\dagger V_{ra}\right) = (2j+1)\delta(a,b) + q^{j(b-a)}\left[\mathrm{e}^{\mathrm{i}(\phi_r - \phi_s)} - 1\right]$$

where the trace is taken on $\varepsilon(j)$ and where $r \in \mathbf{R}$, $s \in \mathbf{R}$ and $a, b = 0, 1, \cdots, 2j$. For r and s such that the condition (4) is satisfied, we have

$$\mathrm{tr}\left(V_{sb}^\dagger V_{ra}\right) = (2j+1)\delta(a,b)$$

with $r \in \mathbf{R}$, $s \in \mathbf{R}$ and $a, b = 0, 1, \cdots, 2j$.

We note that $2j = 1$ is the sole case for which it is possible to find r and s such that $\mathrm{tr}\left(U_s^\dagger U_r\right) = 0$. This explains the peculiarity of the case $2j = 1$.

Result 6. In the case where $2j+1$ is prime, following the works in Refs. 26, 27, 35 and 47, for a given value of r let M be the set of unitary operators

$$M = \{V_{ra} : a = 0, 1, \cdots, 2j\}$$

generated by the two generalized Weyl-Pauli operators U_r and Z. The vectors of the spherical basis $s(j)$ and the eigenvectors of the $2j+1$ operators in M provide a set of of $2j+2$ MUBs for the Hilbert space $\varepsilon(j)$ of dimension $2j+1$.

The derivation of the latter result easily follows by adapting the proof of Theorem 2.3 of Ref. 26.

As an example, we treat the case $j = 1$ with $r = 0$ for which the 12 vectors of the 4 MUBs can be described by a single simple formula. The $2j + 2 = 4$ MUBs consist of the spherical basis $s(1)$ and of the 3 bases (corresponding to $a = 0$, 1 and 2) spanned by the vectors

$$\Psi_a(n_\alpha) = \frac{1}{\sqrt{3}}\left(\omega^{-n_\alpha+a}|1-1\rangle + |10\rangle + \omega^{n_\alpha+a}|11\rangle\right)$$

with $n_\alpha = 0$, 1, 2 and $a = 0$, 1, 2 (as usual, $\omega = \mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$). The vectors $|1m\rangle$ of the spherical basis $s(1)$ are eigenvectors of J_z with the real eigenvalues m. The case $a = 0$ corresponds to the basis $b_0(1)$, the vectors $|1n_\alpha;0\rangle$ of which are eigenvectors of $V_{00} = U_0$ with the complex eigenvalues ω^{-n_α}. More generally, for fixed a (with $a = 0$, 1 or 2), the vectors of the basis $\{\Psi_a(n_\alpha) : n_\alpha = 0, 1 \text{ and } 2\}$ are eigenvectors of the operators V_{0a}:

$$V_{0a}\Psi_a(n_\alpha) = \omega^{-n_\alpha-a}\Psi_a(n_\alpha)$$

As a résumé, by introducing the notation

$$N_{xyz}(x,y,z) \equiv N_{xyz}\left(x|1-1\rangle + y|10\rangle + z|11\rangle\right), \quad N_{xyz} = \frac{1}{\sqrt{|x|^2+|y|^2+|z|^2}}$$

we have the 4 MUBs

$$\begin{aligned}
s(1) &: (1,0,0); (0,1,0); (0,0,1) \\
a = 0 &: \frac{1}{\sqrt{3}}(1,1,1); \frac{1}{\sqrt{3}}(\omega^2,1,\omega); \frac{1}{\sqrt{3}}(\omega,1,\omega^2) \\
a = 1 &: \frac{1}{\sqrt{3}}(\omega,1,\omega); \frac{1}{\sqrt{3}}(1,1,\omega^2); \frac{1}{\sqrt{3}}(\omega^2,1,1) \\
a = 2 &: \frac{1}{\sqrt{3}}(\omega^2,1,\omega^2); \frac{1}{\sqrt{3}}(\omega,1,1); \frac{1}{\sqrt{3}}(1,1,\omega)
\end{aligned}$$

for the space $\varepsilon(1)$. Note that, for fixed a (with $a = 0$, 1 or 2), the basis $\{\Psi_a(n_\alpha) : n_\alpha = 0, 1 \text{ and } 2\}$ spans the regular representation of the cyclic group Z_3. Note also that, the basis $b_0(1)$ corresponds to the three irreducible *vector* representations of Z_3 while the bases for $a = 1$ and $a = 2$ correspond to irreducible *projective* representations of Z_3.

4 Concluding Remarks

The derivation of the usual (i.e., non-deformed) Lie algebra su_2 was achieved in Sec. 2 by adapting the Schwinger trick[45,47] to the case of two deformed oscillator algebras corresponding to a coupled pair of truncated harmonic oscillators. This constitutes an unsual result for Lie algebras. In the context of deformations, we generally start from a Lie algebra, then deform it and finally find a realization in terms of deformed oscillator algebras. Here we started from two q-deformed oscillator algebras from which we derived the non-deformed Lie algebra su_2.

The polar decomposition of the ladder operators of su_2 inherent to our derivation of su_2 led to the scheme $\{J^2, U_r\}$, an alternative to the standard scheme $\{J^2, J_z\}$ of angular momentum theory, a theory familiar to the physicist.

Some of the known results about MUB's were explored in Sec. 3 in the framework of angular momentum theory with a special emphasis on the unitay operator U_r. This shows that the idea of deformations (and possibly Hopf algebras), especially for a deformation parameter taken as a root of unity, could be useful for investigating MUBs. It is also hoped that the so-called Wigner–Racah unit tensors[45] acting on a subspace of constant angular momentum $\varepsilon(j)$ and spanning the Lie algebra of the unitary group U_{2j+1} might be useful for characterizing the operators V_{ra} of Sec. 3. Furthermore, it is worth noting that the parameter r in V_{ra} introduces a further degree of freedom. In this respect, let us mention that, when $2j+1$ is an odd prime number, by replacing r in Eq. (2) by the m-dependent parameter

$$r(m) = -a\frac{(j+m)^2}{jm}, \quad m \neq 0, \quad a = 0, 1, ..., 2j$$

we generate, together with the spherical basis $s(j)$, $2j+2$ MUBs for the space $\varepsilon(j)$. This amounts in last analysis to redefining the operator U_r.

These matters deserve to be further worked out and should be the object of a future work.

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