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GENERALIZED COHERENT STATES FOR POLYNOMIAL WEYL-HEISENBERG ALGEBRAS

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Abstract

It is the aim of this paper to show how to construct *à la* Perelomov and *à la* Barut-Girardello coherent states for a polynomial Weyl-Heisenberg algebra. This algebra depends on r parameters. For some special values of the parameter corresponding to $r = 1$, the algebra covers the cases of the $\mathfrak{su}(1,1)$ algebra, the $\mathfrak{su}(2)$ algebra and the ordinary Weyl-Heisenberg or oscillator algebra. For r arbitrary, the generalized Weyl-Heisenberg algebra admits finite or infinite-dimensional representations depending on the values of the parameters. Coherent states of the Perelomov type are derived in finite and infinite dimensions through a Fock-Bargmann approach based on the use of complex variables. The same approach is applied for deriving coherent states of the Barut-Girardello type in infinite dimension. In contrast, the construction of *à la* Barut-Girardello coherent states in finite dimension can be achieved solely at the price to replace complex variables by generalized Grassmann variables. Finally, some preliminary developments are given for the study of Bargmann functions associated with some of the coherent states obtained in this work.

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1 INTRODUCTION

Coherent states are of paramount importance in physics (e.g., in quantum optics and quantum information theory) and mathematical physics (e.g., in probability theory, applied group theory, path integral formalism and theory of analytic functions) [1-6]. They are generally associated with a quantum system (like the oscillator or the Morse or the Pöschl-Teller systems) or an algebra (like the Weyl-Heisenberg algebra or a Lie algebra). The most well-known coherent states concern the harmonic oscillator system [7]. The coherent states for the $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$ algebras play also an important role in various fields of theoretical and mathematical physics since the pioneer works by Barut and Girardello [8] and by Perelomov [1]. In recent years, as extensions of these well-known examples, generalized coherent states were the object of numerous studies (see for instance [9-16]).

It is the object of the present article to report on a new construction, based on a *à la* Fock-Bargmann approach, of generalized coherent states associated with a polynomial Weyl-Heisenberg algebra. The construction is achieved both in finite and infinite dimensions.

The paper is organized as follows. In Section 2, basics about the coherent states for the harmonic oscillator are briefly reviewed in order to understand which results can or cannot be generalized. Section 3 deals with the study of a polynomial Weyl-Heisenberg algebra which is an extension of the Weyl-Heisenberg algebra for the one-dimensional harmonic oscillator. The main results are contained in Sections 4 and 5; they concern the construction of coherent states of the Perelomov type (Section 4) and of the Barut-Girardello type (Section 5) both in finite and infinite dimensions. Some common properties are given in Section 6. Conclusions and perspectives close this paper in Section 7.

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2 BASICS OF COHERENT STATES

The harmonic oscillator algebra (or usual Weyl-Heisenberg algebra) is spanned by three linear operators, namely, an annihilation operator (a^-), a creation operator (a^+) and a number operator ($N = a^+a^-$) satisfying the relations

$$[a^-, a^+] = I, \quad [N, a^-] = -a^-, \quad [N, a^+] = +a^+, \quad a^+ = (a^-)^\dagger, \quad N = N^\dagger, \quad (1)$$

where I is the identity operator. There three ways to define coherent states for the harmonic oscillator system:

- as eigenvectors $|z\rangle$, $z \in \mathbf{Z}$, of an annihilation operator a^- (\Rightarrow Barut-Girardello type states)
- by acting with displacement operator $\exp(za^+ - \bar{z}a^-)$ on ground state $|0\rangle$ of N (\Rightarrow Perelomov type states)
- by minimizing the uncertainty relation for the position and momentum operators associated with a^- and a^+ (\Rightarrow Roberston-Schrödinger type states).

The three ways lead to the same coherent states, a result that is not true for other dynamical systems. The expression for the coherent states of the harmonic oscillator, the so-called Glauber states [7], reads (up to a normalization factor)

$$|z\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} z^n |n\rangle$$

in terms of eigenvectors

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^+)^n |0\rangle$$

of operator N .

The situation is different for other dynamical systems or algebras. This is well-known for the $\text{su}(1, 1)$ algebra: the Perelomov states obtained through the action of displacement operator $\exp(zK^+ - \bar{z}K^-)$ on ground state $|k, 0\rangle$ of the positive discrete series representation of $\text{su}(1, 1)$ are different from the Barut-Girardello states arising from eigenvalue equation $K_-|z\rangle = z|z\rangle$ (K^+ and K_- are the two ladder operators of $\text{su}(1, 1)$). In the $\text{su}(2)$ case, it is possible to define Perelomov states owing to displacement operator $\exp(zJ^+ - \bar{z}J^-)$ acting on ground state $|j, -j\rangle$ of the $2j + 1$ -dimensional representation of $\text{su}(2)$ (J^+ and J_- are the two ladder operators of $\text{su}(2)$); however, it is not possible to define Barut-Girardello states $|z\rangle$, $z \in \mathbf{C}$, for $\text{su}(2)$ as eigenstates of J_- .

3 GENERALIZED WEYL-HEISENBERG ALGEBRA

3.1 Polynomial Weyl-Heisenberg algebra

Following many works on possible extensions of the usual Weyl-Heisenberg algebras [17-27], let us consider the algebra spanned by an annihilation operator (a^-), a creation operator (a^+) and a number operator ($N \neq a^+a^-$) satisfying the commutation relations

$$[a^-, a^+] = G(N), \quad [N, a^-] = -a^-, \quad [N, a^+] = +a^+, \quad (2)$$

with

$$a^+ = (a^-)^\dagger, \quad N = N^\dagger, \quad G(N) = F(N + 1) - F(N), \quad (3)$$

where the F structure function is defined by

$$F(N) = N \prod_{i=1}^r [I + \kappa_i(N - I)], \quad \kappa_i \in \mathbf{R} \quad (i = 1, 2, \dots, r). \quad (4)$$

Equations (2)–(4) constitute a polynomial extension of the usual Weyl-Heisenberg algebra defined by (1). This polynomial Weyl-Heisenberg algebra, denoted as $\mathcal{A}_{\{\kappa\}}$, depends on r real parameters. Of course, other choices for $F(N)$ lead to other generalized Weyl-Heisenberg algebras.

Three interesting particular cases for $F(N)$ correspond to

$$\kappa_1 = \kappa, \quad \kappa_2 = \kappa_3 = \dots = \kappa_r = 0.$$

Then, the special case where $\kappa = 0$ ($F(N) = N \Rightarrow G(N) = I$) corresponds to the usual harmonic oscillator system (described by the h_4 usual Weyl-Heisenberg algebra). Furthermore, the cases $\kappa > 0$ and $\kappa < 0$ describe the Pöschl-Teller system (described by the $\text{su}(1, 1)$ algebra) and the Morse system (described by the $\text{su}(2)$ algebra), respectively [25, 27, 28].

3.2 Representation of the polynomial Weyl-Heisenberg algebra

Going back to the general case, since the $\mathcal{A}_{\{\kappa\}}$ algebra is an extension of the usual oscillator algebra, we may hope to find a representation of $\mathcal{A}_{\{\kappa\}}$ which extends that of h_4 . Indeed, it is easy to check that the actions

$$a^-|n\rangle = \sqrt{F(n)}e^{+i[F(n)-F(n-1)]\varphi}|n-1\rangle, \quad a^-|0\rangle = 0, \quad (5)$$

$$a^+|n\rangle = \sqrt{F(n+1)}e^{-i[F(n+1)-F(n)]\varphi}|n+1\rangle, \quad N|n\rangle = n|n\rangle \quad (6)$$

(on the Hilbert space spanned by the eigenvectors of N) formally define a representation of $\mathcal{A}_{\{\kappa\}}$. The φ parameter is a real parameter which is generally taken to be 0 in developments concerning the harmonic oscillator; we shall see that this parameter is essential to ensure temporal stability of coherent states. Note that

$$a^+a^- = F(N),$$

a relation that generalizes $N = a^+a^-$ for the harmonic oscillator and gives a significance to the F function: $F(N)$ can be considered as the Hamiltonian for a quantum system.

We may now ask what is the dimension of the representation (Fock-Hilbert) space generated by the orthonormal set $\{|n\rangle : n \text{ ranging}\}$? The dimension of the representation of $\mathcal{A}_{\{\kappa\}}$ afforded by (5) and (6) is controlled by the positiveness of:

$$F(n) = n \prod_{i=1}^r [1 + \kappa_i(n-1)] \geq 0.$$

We shall limit ourselves here to two cases.

- $\kappa_i \geq 0$ ($i = 1, 2, \dots, r$): there is no limit to the number of states $|n\rangle$ and the representation is infinite-dimensional so that the Fock-Hilbert space is generated by $\{|n\rangle : n \in \mathbf{N}\}$,
- $\kappa_1 < 0, \kappa_i \geq 0$ ($i = 2, 3, \dots, r$): the number of states $|n\rangle$ is limited and the representation has dimension d with

$$d = 1 - \frac{1}{\kappa_1}, \quad -1/\kappa_1 \in \mathbf{N}^* \quad \Rightarrow \quad F(n) = n \frac{d-n}{d-1} \prod_{i=2}^r [1 + \kappa_i(n-1)],$$

so that the Fock-Hilbert space is generated by $\{|n\rangle : n = 0, 1, \dots, d-1\}$.

In the finite-dimensional case, two further conditions are verified. Indeed, it can be shown that

$$a^+|d-1\rangle = 0, \quad (a^-)^d = (a^+)^d = 0,$$

two relations that generalize the conditions for k -fermions [11, 12] (the $d = k = 2$ case corresponds to ordinary fermions).

3.3 Truncated polynomial Weyl-Heisenberg algebra

In the infinite-dimensional case, it can be useful to truncate the representation space to a subspace of dimension s (for defining a unitary phase operator or for perturbation theory purposes). This can

be achieved via the Pegg-Barnett trick developed for the h_4 oscillator algebra [29]. This amounts to replace the a^\pm operators by

$$a^\pm(s) = a^\pm - \sum_{n=s}^{\infty} \sqrt{F(n)} e^{\mp i[F(n)-F(n-1)]\varphi} |n - \frac{1}{2} \pm \frac{1}{2}\rangle \langle n - \frac{1}{2} \mp \frac{1}{2}|.$$

Therefore, we pass from the $\mathcal{A}_{\{\kappa\}}$ algebra to the $\mathcal{A}_{\{\kappa,s\}}$ truncated algebra defined by

$$[a^-(s), a^+(s)] = G_s(N) - F(s)|s-1\rangle\langle s-1|, \quad [N, a^\pm(s)] = \pm a^\pm(s),$$

with

$$a^+(s) = (a^-(s))^\dagger, \quad N = N^\dagger, \quad G_s(N) = \sum_{n=0}^{s-1} [F(n+1) - F(n)] |n\rangle\langle n|.$$

Thus, the results derived for a Weyl-Heisenberg algebra with a representation of dimension d can be applied to a $\mathcal{A}_{\{\kappa,s\}}$ truncated algebra arising from another Weyl-Heisenberg algebra with an infinite-dimensional representation.

4 PERELOMOV TYPE COHERENT STATES

The derivation of *à la* Perelomov coherent states for an arbitrary $\mathcal{A}_{\{\kappa\}}$ algebra from the action of a displacement operator on state $|0\rangle$ is very difficult because commutator $[a^-, a^+]$ differs from the identity operator. Consequently, we shall adopt a more simple strategy based on the use of a Fock-Bargmann space associated with $\mathcal{A}_{\{\kappa\}}$. This strategy can be summed up as follows.

Let us look for states in the form

$$|z, \varphi\rangle = \sum_n \bar{a}_n z^n |n\rangle, \quad a_n \in \mathbf{C}, \quad z \in \mathbf{C}, \quad (7)$$

where the sum on n is finite or infinite according to as $\mathcal{A}_{\{\kappa\}}$ admits a finite- or infinite-dimensional representation. The a_n coefficients can then be determined from the correspondence rules

$$|n\rangle \longrightarrow a_n z^n, \quad a^- \longrightarrow \frac{d}{dz} \quad (8)$$

applied to relations (5) and (6). The convergence of the $|z, \varphi\rangle$ states so-obtained should be checked as well as their existence as Perelomov type coherent states.

4.1 The infinite case

The strategy just described leads to the following recurrence relation

$$na_n = \sqrt{F(n)} e^{+i[F(n)-F(n-1)]\varphi} a_{n-1}, \quad (9)$$

which can be iterated to give

$$a_n = \frac{\sqrt{F(n)!}}{n!} e^{+iF(n)\varphi}, \quad (10)$$

(by taking $a_0 = 1$). In Eq. (10), the generalized factorials are defined by

$$F(0)! = 1, \quad F(n)! = F(1)F(2) \dots F(n).$$

This yields the following result.

Result 1. In infinite dimension, the states

$$|z, \varphi\rangle = \sum_{n=0}^{\infty} \frac{\sqrt{F(n)!}}{n!} z^n e^{-iF(n)\varphi} |n\rangle$$

exist only for $r = 1$ in the disk $\{z \in \mathbf{C} : |z| < 1/\sqrt{\kappa_1}\}$. They satisfy $|z, \varphi\rangle = \exp(za^+)|0\rangle$ and are thus coherent states in the Perelomov sense.

We note that the restriction on r comes from the fact that the $|z, \varphi\rangle$ states cannot be normalized if $r \geq 2$.

Example 1. Let us examine the case where $r = 1$ and $\kappa_1 = 1/\ell$ with $\ell \in \mathbf{N}^*$. The corresponding coherent states read

$$|z, \varphi\rangle = \sum_{n=0}^{\infty} \sqrt{\frac{1}{n!} \frac{(\ell - 1 + n)!}{\ell^n (\ell - 1)!}} z^n e^{-iF(n)\varphi} |n\rangle.$$

Note that the $\ell \rightarrow \infty$ limit corresponds to the harmonic oscillator.

4.2 The finite case

In this case, recurrence relation (9) is valid. However, there is no restriction on r for normalization purposes. We are thus left with Result 2.

Result 2. In finite dimension ($\dim = d$ or s), the states

$$|z, \varphi\rangle = \sum_{n=0}^{\dim-1} \frac{\sqrt{F(n)!}}{n!} z^n e^{-iF(n)\varphi} |n\rangle$$

exist for any value of r and any z in \mathbf{C} . They satisfy $|z, \varphi\rangle = \exp(za^+)|0\rangle$ and are thus coherent states in the Perelomov sense.

Example 2. For $r = 1$ and $\dim = d$ (the $\mathcal{A}_{\{\kappa\}}$ algebra has a representation of dimension d), $|z, \varphi\rangle$ reads

$$|z, \varphi\rangle = \sum_{n=0}^{d-1} \sqrt{\frac{1}{n!} \frac{(d-1)!}{(d-1)^n (d-1-n)!}} z^n e^{-iF(n)\varphi} |n\rangle.$$

Note that the $d \rightarrow \infty$ limit corresponds to the harmonic oscillator.

5 BARUT-GIRARDELLO TYPE COHERENT STATES

A strategy similar to that used for coherent states of the Perelomov type can be set up for the determination of Barut-Girardello type coherent states associated with $\mathcal{A}_{\{\kappa\}}$. It consists in looking for states in the form given by (7) and in replacing (8) by

$$|n\rangle \longrightarrow a_n z^n, \quad a^+ \longrightarrow z. \quad (11)$$

5.1 The infinite case

By introducing Eq. (11) in (5) and (6), we get the recurrence relation

$$a_n = \sqrt{F(n+1)} e^{-i[F(n+1)-F(n)]\varphi} a_{n+1}$$

which admits the solution

$$a_n = \frac{1}{\sqrt{F(n)!}} e^{+iF(n)\varphi}$$

(we take $a_0 = 1$). As a conclusion, we have the next result.

Result 3. In infinite dimension, the states

$$|z, \varphi\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{F(n)!}} z^n e^{-iF(n)\varphi} |n\rangle$$

exist for any value of r in the whole complex plane \mathbf{C} . They satisfy $a^-|z, \varphi\rangle = z|z, \varphi\rangle$ and are thus coherent states in the Barut-Girardello sense.

Example 3. In the special case where $r = 1$ and $\kappa_1 = 1/\ell$ with $\ell \in \mathbf{N}^*$, we have

$$|z, \varphi\rangle = \sum_{n=0}^{\infty} \sqrt{\frac{1}{n!} \frac{\ell^n (\ell - 1)!}{(\ell - 1 + n)!}} z^n e^{-iF(n)\varphi} |n\rangle$$

Note that the $\ell \rightarrow \infty$ limit corresponds to the harmonic oscillator.

5.2 The finite case

The situation is quite new in finite dimension ($\dim = d$ or s). Indeed, the strategy applied in the last subsection to the infinite case requires that either the $|z, \varphi\rangle$ states are identically 0 or $z^{\dim} = 0$. Therefore, there is only the trivial solution if z is a complex variable. However, if z is replaced by a Grassmann variable, θ , of order \dim (i.e., $\theta^{\dim} = 0$), we obtain the following result.

Result 4. In finite dimension ($\dim = d$ or s), there are no Barut-Girardello coherent states for $z \in \mathbf{C}$. However, Barut-Girardello coherent states exist for

$$z \rightarrow \theta = \text{Grassmann variable with } \theta^{\dim} = 0.$$

They are given by

$$|\theta, \varphi\rangle = \sum_{n=0}^{\dim-1} \frac{1}{\sqrt{F(n)!}} \theta^n e^{-iF(n)\varphi} |n\rangle$$

for any value of r and satisfy $a^-|\theta, \varphi\rangle = \theta|\theta, \varphi\rangle$.

It should be noted that when $\dim \rightarrow \infty$ and $\theta \rightarrow z$, we get back the coherent states for the harmonic oscillator.

Example 4. For $r = 1$ and $\dim = d = 2$, we have the states

$$|\theta, \varphi\rangle = |0\rangle + \theta e^{-i\varphi} |1\rangle,$$

which for $\varphi = 0$ coincide with the coherent states for the fermionic oscillator [30] (of interest for qubits).

6 COMMON PROPERTIES

The Perelomov and Barut-Girardello coherent states derived above share some common properties which can be summarized as follows. (Further details shall be published elsewhere [31].)

- They are continuous in the variables φ and z or θ .
- They are stable under time evolution, i.e.,

$$e^{-iHt}|z \text{ or } \theta, \varphi\rangle = |z \text{ or } \theta, \varphi + t\rangle, \quad H = F(N) = a^+ a^-.$$

- They are normalizable but not orthogonal.
- They satisfy overcompleteness relations, i.e.,

$$\int d\mu(|z|)|z, \varphi\rangle\langle z, \varphi| = \sum_{n=0}^{\infty} |n\rangle\langle n| \text{ or } \int d\mu(|z|)|z \text{ or } \theta, \varphi\rangle\langle z \text{ or } \theta, \varphi| = \sum_{n=0}^{dim-1} |n\rangle\langle n|$$

in infinite or finite dimension, where the $d\mu$ measures are given in [31] in terms of special functions (complex variable case) or generalized Berezin calculus (Grassmann variable case).

7 SUMMARY, CONCLUSIONS AND PERSPECTIVES

We focused in this work on a r -parameter polynomial Weyl-Heisenberg algebra, $\mathcal{A}_{\{\kappa\}}$, that generalizes the oscillator algebra. We showed that this algebra admits infinite- or finite-dimensional representations depending on the value of the parameters. In addition, $\mathcal{A}_{\{\kappa\}}$ can describe dynamical quantum systems with nonlinear (in n) spectra and can serve as a framework for generating phase operators, phase states and mutually unbiased bases (for $r = 1$, see [27, 32]). We developed a simple and straightforward derivation, in a Fock-Bargmann approach, of Perelomov and Barut-Girardello coherent states for finite- and infinite-dimensional representations of $\mathcal{A}_{\{\kappa\}}$. It is to be noted that our construction of Barut-Girardello coherent states in dimension d in terms of Grassmann variables establishes a link with k -fermions [11, 12] which are objects interpolating between fermions ($k = d = 2$) and bosons ($k = d \rightarrow \infty$).

As open questions and perspectives, we can mention: a probabilistic interpretation and the study of Bargmann functions associated with some of the coherent states obtained in this work. In this respect, we close with some preliminary developments concerning Bargmann functions for the Barut-Girardello type coherent states in infinite dimension.

We shall restrict ourselves to the case

$$\kappa_i = 1/\ell_i, \quad \ell_i \in \mathbf{N}^* \quad (i = 1, 2, \dots, r).$$

Then, the Barut-Girardello coherent states given in Result 3 can be normalized as

$$|z, \varphi\rangle = \mathcal{N}^{-1} \sum_{n=0}^{\infty} \frac{1}{\sqrt{F(n)!}} z^n e^{-iF(n)\varphi} |n\rangle, \quad |\mathcal{N}|^2 = {}_0F_r(\ell_1, \ell_2, \dots, \ell_r; \ell_1 \ell_2 \dots \ell_r |z|^2).$$

With respect to these coherent states, the vector

$$|f\rangle = \sum_{n=0}^{\infty} f_n |n\rangle, \quad \sum_{n=0}^{\infty} |f_n|^2 < \infty$$

can be represented by the f_φ analytical function defined by

$$f_\varphi(z) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{F(n)!}} z^n e^{-iF(n)\varphi} f_n.$$

Let us recall that the growth of an arbitrary entire series, say $f(z) = \sum_{n=0}^{\infty} c_n z^n$, is described by means of two nonnegative numbers: order ρ and type σ given by [33]

$$\rho = \lim_{n \rightarrow \infty} \left(-n \frac{\log n}{\log |c_n|} \right), \quad \sigma = \frac{1}{e\rho} \lim_{n \rightarrow \infty} \left(n |c_n|^{\frac{\rho}{n}} \right).$$

This allows to classify entire functions according to their growth as $|z| \rightarrow \infty$: the maximum modulus $M(R)$ of $f(z)$ for $|z| = R$ behaves like

$$M(R) \sim \exp(\sigma |z|^\rho)$$

as R goes to infinity. It is simple to verify (through the use of Schwarz inequality) that

$$|f_\varphi(z)| \leq |\mathcal{N}|$$

and, using arguments similar to those in [9], it can be shown that order ρ and type σ of the f_φ function are

$$\rho = \frac{2}{1+r}, \quad \sigma = \frac{1+r}{2} (\ell_1 \ell_2 \dots \ell_r)^{\frac{1}{1+r}}. \quad (12)$$

It is interesting to note that the ρ order of the Bargmann functions associated with Barut-Girardello coherent states decreases as r increases. In the particular case where $r = 1$ and $\ell_1 = 1$, Eq. (12) is in agreement with the result for Example B of [34] which corresponds in our notation to $F(n)! = (n!)^2$. (Of course, the standard harmonic oscillator case is trivial and can be recovered by setting $F(n) = n$.) Finally, let us mention that Eq. (12) can also be derived from the behavior of the measure for the Barut-Girardello states (see [31]) by using a method similar to that of [34].

References

- [1] J. R. Klauder and Bo-S. Skagerstam, *Coherent States – Applications in Physics and Mathematical Physics*, World Scientific, Singapore, 1985.
- [2] A. Perelomov, *Generalized Coherent States and Their Applications*, Springer, Berlin, 1986.
- [3] S. T. Ali, J.-P. Antoine, and J.-P. Gazeau, *Coherent States, Wavelets, and Their Generalizations*, Springer, Berlin, 2000.
- [4] K. Fujii, “Introduction to Coherent States and Quantum Information Theory”, arXiv:0112090 [quant-ph] (2002).
- [5] G. Drăgănescu, *Stări coerente – aplicații*, Editura Politehnica, Timișoara, 2008.
- [6] J.-P. Gazeau, *Coherent States in Quantum Physics*, WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim, 2009.
- [7] R. J. Glauber, *Phys. Rev.* **130**, 2529 (1963); *ibid.* **131**, 2766 (1963).
- [8] A. O. Barut and L. Girardello, *Commun. Math. Phys.* **21**, 41 (1971).
- [9] D. J. Fernández, V. Hussin, and L. M. Nieto, *J. Phys. A: Math. Gen.* **27**, 3547 (1994).
- [10] J.R. Klauder, *Ann. Phys. (NY)* **237**, 147 (1996).
- [11] M. Daoud, Y. Hassouni, and M. Kibler, “The k -fermions as objects interpolating between fermions and bosons,” in *Symmetries in Science X*, edited by B. Gruber and M. Ramek, Plenum Press, New York, 1998.
- [12] M. Daoud, Y. Hassouni, and M. Kibler, *Phys. Atom. Nuclei* **61**, 1821 (1998).
- [13] J.-P. Gazeau and J. R. Klauder, *J. Phys. A: Math. Gen.* **32**, 123 (1999).
- [14] J. M. Sixdeniers, K. A. Penson, and A. I. Solomon, *J. Phys. A: Math. Gen.* **32**, 7543 (1999).
- [15] C. Quesne, *Ann. Phys. (NY)* **293**, 147 (2001).
- [16] J. R. Klauder, K. A. Penson, and J. M. Sixdeniers, *Phys. Rev. A* **64**, 013817 (2001).
- [17] C. Daskaloyannis, *J. Phys. A: Math. Gen.* **24**, L789 (1991).
- [18] C. Quesne and N. Vansteenkiste, *J. Phys. A: Math. Gen.* **28**, 7019 (1995); *Helv. Phys. Acta* **69**, 141 (1996).
- [19] J. Katriel and C. Quesne, *J. Math. Phys.* **37**, 1650 (1996).
- [20] V. I. Man’ko, G. Marmo, F. Zaccaria, and E. C. G. Sudarshan, *Phys. Scr.* **55**, 528 (1997).
- [21] C. D. J. Fernández and V. Hussin, *J. Phys. A: Math. Gen.* **32**, 3603 (1999).

- [22] A. H. El Kinani and M. Daoud, *Int. J. Modern. Phys. B* **15**, 2465 (2001); *J. Math. Phys.* **43**, 714 (2002).
- [23] M. Daoud and M. Kibler, *Phys. Part. Nucl.* **33**, S43 (2002).
- [24] J. M. Carballo, C. D. J. Fernández, J. Negro, and L. M. Nieto, *J. Phys. A: Math. Gen.* **37**, 10349 (2004).
- [25] M. Daoud and M. R. Kibler, *J. Math. Phys.* **47**, 122108 (2006).
- [26] A. M. Gavrilik and A. P. Rebesh, *J. Phys. A: Math. Theor.* **43**, 095203 (2010).
- [27] M. Daoud and M. R. Kibler, *J. Phys. A: Math. Theor.* **43**, 115303 (2010).
- [28] N. M. Atakishiyev, M. R. Kibler, and K. B. Wolf, *Symmetry* **2**, 1461 (2010).
- [29] D. T. Pegg and S. M. Barnett, *Phys. Rev. A* **39**, 1665 (1989).
- [30] Y. Bérubé-Lauzière and V. Hussin, *J. Phys. A: Math. Gen.* **26**, 6271 (1993).
- [31] M. Daoud and M. R. Kibler, “Bosonic and k -fermionic coherent states for a class of polynomial Weyl-Heisenberg algebras”, to be published in *J. Phys. A: Math. Theor.*, arXiv:1110.4799 [quant-ph].
- [32] M. Daoud and M. R. Kibler, *J. Math. Phys.* **52**, 082101 (2011).
- [33] B. Ya. Levin, *Lectures on Entire Functions*, Rhode Island, American Math. Soc. (1996).
- [34] A. Vourdas, K. A. Penson, G. H. E. Duchamp, and A. I. Solomon, “Generalized Bargmann functions, their growth and von Neumann lattices”, arXiv:1111.0889 [quant-ph] (2011).