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# Linking partial and quasi dynamical symmetries in rotational nuclei

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We establish a link between two distinct symmetry concepts, partial dynamical symmetry (PDS) and quasi dynamical symmetry (QDS). The connection is illustrated in the framework of the interacting boson model of nuclei. Quantum-number fluctuations reveal a previously unrecognized region of Hamiltonians that have both  $O(6)$  PDS (purity) and  $SU(3)$  QDS (coherence) in the ground band. Many rare-earth nuclei can be identified approximately satisfying both symmetry requirements.

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Understanding the structure and dynamics of complex many-body systems can often be obtained from the observation and analysis of symmetries. Symmetry considerations are particularly significant for addressing a key question in such systems, namely, how do simple features emerge within a complicated environment. A notable example is the collective behavior of nuclei which stems from the complex interactions among the constituent nucleons. Despite the complex nature of the low-energy effective forces at work and the large number of participating particles, collective nuclei give rise to strikingly regular excitation spectra, signaling the presence of underlying symmetries [1]. The theme of “simplicity out of complexity” and the understanding of simple emergent behavior are major challenges facing the study of almost any many-body system, from atomic nuclei to nanoscale and macroscopic systems [2].

Although, usually, a many-body Hamiltonian does not conform to a dynamical symmetry (DS) limit [3], the possibility exists that certain symmetries are obeyed by only a subset of its eigenstates. This situation, referred to as partial dynamical symmetry (PDS) [4], was shown to be relevant to specific nuclei and molecules [4–13]. In parallel, the notion of quasi dynamical symmetry (QDS) was introduced and discussed in the context of nuclear models [14–21]. While QDS can be defined mathematically in terms of embedded representations [22, 23], its physical meaning is that several observables associated with a particular subset of eigenstates, may be consistent with a certain symmetry which in fact is broken in the Hamiltonian. This typically occurs for a Hamiltonian transitional between two DS limits which retains, for a certain range of its parameters, the characteristics of one of those limits. This “apparent” symmetry is due to a coherent mixing of representations in selected states, imprinting an adiabatic motion and increased regularity [19–21].

PDS and QDS are applicable to any many-body problem (bosonic and fermionic) endowed with an algebraic structure. They play a role in diverse phenomena including nuclear and molecular spectroscopy, quantum phase

transitions and mixed regular and chaotic dynamics. In this Letter, a hitherto unnoticed link is established between these two different symmetry concepts and it is shown that coherent mixing of one symmetry (QDS) can result in the partial conservation of a different, incompatible symmetry (PDS). An empirical manifestation of such a linkage is presented.

Algebraic models provide a convenient framework for exploring the role of symmetries [24]. One such framework is the interacting boson model (IBM) [25], which has been widely used to describe quadrupole collective states in nuclei in terms of  $N$  monopole ( $s^\dagger$ ) and quadrupole ( $d^\dagger$ ) bosons, representing valence nucleon pairs. The model has  $U(6)$  as a spectrum generating algebra and exhibits three DS limits, associated with chains of nested subalgebras, starting with  $U(5)$ ,  $O(6)$ , and  $SU(3)$ , respectively. These solvable limits correspond to known benchmarks of the geometric description of nuclei [26], involving vibrational [ $U(5)$ ],  $\gamma$ -soft [ $O(6)$ ], and rotational [ $SU(3)$ ] types of dynamics. In what follows we employ the IBM as test ground for connecting the PDS and QDS notions. The particular example considered, namely,  $SU(3)$  QDS as an emanation of  $O(6)$  PDS, is shown to have approximate validity in many deformed rare-earth nuclei.

One particularly successful approach within the IBM is the extended consistent-Q formalism (ECQF) [27, 28], which is frequently used for the interpretation and classification of nuclear data. It uses the same quadrupole operator,  $\hat{Q}^x = d^\dagger s + s^\dagger \tilde{d} + \chi (d^\dagger \tilde{d})^{(2)}$ , in the  $E2$  transition operator and in the Hamiltonian, the latter being written as

$$\hat{H}_{\text{ECQF}} = \omega \left[ (1 - \xi) \hat{n}_d - \frac{\xi}{4N} \hat{Q}^x \cdot \hat{Q}^x \right], \quad (1)$$

where  $\hat{n}_d$  is the d-boson number operator,  $\hat{Q}^x \cdot \hat{Q}^x$  is the quadrupole interaction, and the dot implies a scalar product. The parameters  $\omega$ ,  $\xi$ , and  $\chi$  are fitted to empirical data or calculated microscopically if possible;  $\xi$  and  $\chi$  are the sole structural parameters of the model since  $\omega$

is a scaling factor. The parameter ranges  $0 \leq \xi \leq 1$  and  $-\frac{\sqrt{7}}{2} \leq \chi \leq 0$  interpolate between the  $U(5)$ ,  $O(6)$ , and  $SU(3)$  DS limits, which are reached for  $(\xi, \chi) = (0, \chi)$ ,  $(1, 0)$ , and  $(1, -\frac{\sqrt{7}}{2})$ , respectively. It is customary to represent the parameter space by a symmetry triangle [29], whose vertices correspond to these limits. The ECQF has been used extensively for the description of nuclear properties (see, *e.g.*, Ref. [30]) and it was found that rotational nuclei are best described by ECQF parameters in the interior of the triangle, away from the naively expected  $SU(3)$  DS limit. The  $SU(3)$  mixing was found to be strong and coherent, *i.e.*, the same for all rotational states in a band, exemplifying a  $SU(3)$ -QDS [19–21]. In what follows we examine the  $O(6)$  symmetry properties of ground-band states in such nuclei, in the rare-earth region, using the ECQF of the IBM.

The  $O(6)$  DS basis states are specified by quantum numbers  $N$ ,  $\sigma$ ,  $\tau$ , and  $L$ , related to the algebras in the chain  $U(6) \supset O(6) \supset O(5) \supset O(3)$  [31]. Given an eigenstate  $\Psi$  of the ECQF Hamiltonian (1), its expansion in the  $O(6)$  basis reads

$$|\Psi(\xi, \chi)\rangle = \sum_i \alpha_i(\xi, \chi) |N, \sigma_i, \tau_i, L\rangle, \quad (2)$$

where the sum is over all basis states and, for simplicity, the dependence of  $\Psi$  and  $\alpha_i$  on the boson number  $N$  and the angular momentum  $L$  is suppressed. The degree of  $O(6)$  symmetry of the state  $\Psi$  is inferred from the fluctuations in  $\sigma$  which can be calculated as

$$\Delta\sigma_\Psi = \sqrt{\sum_i \alpha_i^2 \sigma_i^2 - \left(\sum_i \alpha_i^2 \sigma_i\right)^2}. \quad (3)$$

If  $\Psi$  carries an exact  $O(6)$  quantum number,  $\sigma$  fluctuations are zero,  $\Delta\sigma_\Psi = 0$ . If  $\Psi$  contains basis states with different  $O(6)$  quantum numbers, then  $\Delta\sigma_\Psi > 0$ , indicating that the  $O(6)$  symmetry is broken. Note that  $\Delta\sigma_\Psi$  also vanishes for a state with a mixture of components with the same  $\sigma$  but different  $O(5)$  quantum numbers  $\tau$ , corresponding to a  $\Psi$  with good  $O(6)$  but mixed  $O(5)$  character. This method of quantifying the  $O(6)$  purity of states has already been applied to  $^{124}\text{Xe}$  [32]. Also,  $\Delta\sigma_\Psi$  has the same physical content as wave-function entropy which, upon averaging over all eigenstates, discloses the global DS content of a given Hamiltonian [33]. We examine here the fluctuations  $\Delta\sigma_\Psi$  for the entire parameter space of the ECQF Hamiltonian (1) for values of  $N$  up to 60, using the ArbModel code [34].

Results of this calculation for the ground state,  $\Psi = 0_{\text{gs}}^+$ , with  $N = 14$  and parameters  $\xi \in [0, 1]$ ,  $\chi \in [-\frac{\sqrt{7}}{2}, 0]$ , are shown in Fig. 1. At the  $O(6)$  DS limit ( $\xi = 1$ ,  $\chi = 0$ )  $\Delta\sigma_{\text{gs}}$  vanishes per construction whereas it is greater than zero for all other parameter pairs. Towards the  $U(5)$  DS limit ( $\xi = 0$ ), the fluctuations reach a saturation value of  $\Delta\sigma_{\text{gs}} \approx 2.47$ . At the  $SU(3)$  DS limit ( $\xi = 1$ ,  $\chi = -\frac{\sqrt{7}}{2}$ )

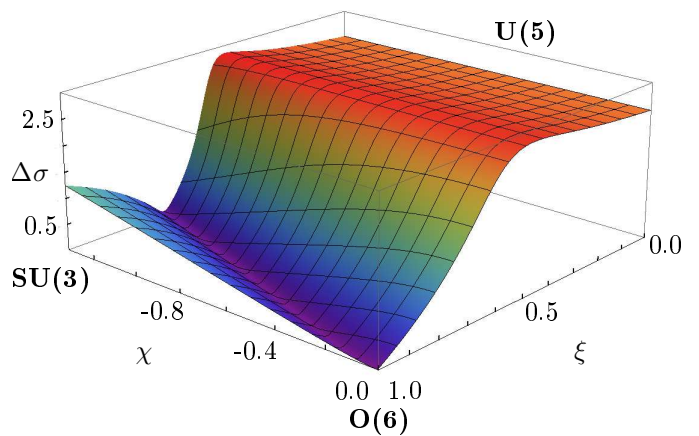


FIG. 1. (Color online) Ground-state fluctuations  $\Delta\sigma_{\text{gs}}$  (3) for the ECQF Hamiltonian (1) with  $N = 14$  bosons. The fluctuations vanish at the  $O(6)$  DS limit, saturate towards the  $U(5)$  DS limit, and are of the order  $10^{-2}$  in the valley.

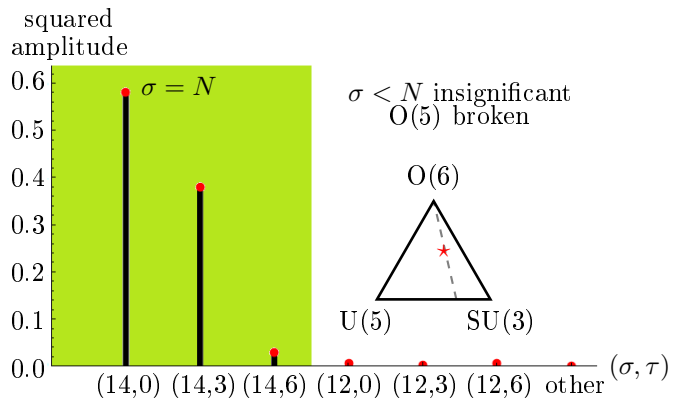


FIG. 2. (Color online) Squared amplitudes  $\alpha_i^2$  in the expansion (2) of the  $0_{\text{gs}}^+$  ground state of the ECQF Hamiltonian (1) for  $\xi = 0.84$  and  $\chi = -0.53$  (indicated by the red star in the symmetry triangle and appropriate for  $^{160}\text{Gd}$ ).

the fluctuations are  $\Delta\sigma_{\text{gs}} \approx 1.25$ . In both cases the  $O(6)$  symmetry is completely dissolved as measured by  $\sigma_{\text{crit}} = 0.849$  [32]. Surprisingly, there is a previously unrecognized valley of almost vanishing  $\Delta\sigma_{\text{gs}}$  values, two orders of magnitude lower than at saturation. This region represents a parameter range of the IBM, outside the  $O(6)$  DS limit, where the ground-state wave function exhibits an exceptionally high degree of purity with respect to the  $O(6)$  quantum number  $\sigma$ .

The ground-state wave functions in the valley of low  $\Delta\sigma_{\text{gs}}$  can be analyzed with the help of the  $O(6)$  decomposition (2). At the  $O(6)$  DS limit only one  $O(6)$  basis state, with  $\sigma = N$  and  $\tau = 0$  contributes, while outside this limit the wave function consists of multiple  $O(6)$  basis states. Investigation of the wave function for parameter combinations inside the valley reveals an overwhelming dominance of the  $O(6)$  basis states with  $\sigma = N$ . This is seen in Fig. 2 for the ground-state wave function of

the ECQF Hamiltonian (1) at  $\xi = 0.84$  and  $\chi = -0.53$  with  $N = 14$ , parameter values that apply to the nucleus  $^{160}\text{Gd}$  discussed below. The  $\sigma = N$  states comprise more than 99% of the ground-state wave function at the bottom of the valley and their dominance causes  $\Delta\sigma_{\text{gs}}$  to be small. Furthermore, it is evident that at the same time the  $O(5)$  symmetry is broken, as basis states with different quantum number  $\tau$  contribute significantly to the wave function. Consequently, the valley can be identified as an entire region in the symmetry triangle with an approximate PDS of type III [4], which means that some of the eigenstates exhibit some of the symmetries. Outside this valley the ground state is a mixture of several  $\sigma$  values and  $\Delta\sigma_{\text{gs}}$  increases. In the  $SU(3)$  DS limit the  $\sigma = N$  components constitute 67% of the wave function and in the  $U(5)$  DS limit and throughout the plateau of saturated  $\Delta\sigma_{\text{gs}}$  this contribution drops below 1%. This region of approximate ground-state  $O(6)$  symmetry is similar to the previously established “arc of regularity” [35] which is a region of reduced mixing inside the IBM parameter space attributed to an approximate  $SU(3)$  symmetry [36].

An argument for the existence of the valley of ground-state  $O(6)$  symmetry can be given in terms of the following Hamiltonian [7]:

$$\hat{H}_M = -\hat{C}_{O(6)} + \hat{N}(\hat{N} + 4) + 2\alpha\hat{C}_{O(5)} - \alpha\hat{C}_{O(3)} + 2\alpha\hat{n}_d(\hat{N} - 2) + \sqrt{14}\alpha(d^\dagger s + s^\dagger \tilde{d}) \cdot (d^\dagger \tilde{d})^{(2)}, \quad (4)$$

where  $\hat{C}_G$  denotes the quadratic Casimir operator of the group  $G$  [25],  $\hat{N}$  is the total boson number operator, and  $\alpha$  is a parameter. The Hamiltonian (4) generates a PDS of type III [4]. For  $\alpha = 0$ ,  $\hat{H}_M$  has exact  $O(6)$  symmetry whereas for  $\alpha > 0$  the last two terms introduce  $O(6)$ -symmetry breaking. However, the yrast states of this Hamiltonian, projected from the IBM intrinsic state with intrinsic variables [37]  $\beta = 1$  and  $\gamma = 0$ , keep exact  $O(6)$  symmetry ( $\sigma = N$ ) but break the  $O(5)$  symmetry (mixed  $\tau$ ) for all values of  $\alpha > 0$  [7]. Interestingly, although  $\hat{H}_M$  differs from  $\hat{H}_{\text{ECQF}}$ , the overlap between their  $0_{\text{gs}}^+$  ground states maximizes (more than 99%) in extended regions of  $(\xi, \chi)$  inside the valley of low  $\Delta\sigma_{\text{gs}}$ . This suggests that the  $(\beta = 1, \gamma = 0)$  intrinsic state provides a good approximation, in a variational sense, to the ground band of  $\hat{H}_{\text{ECQF}}$  along the valley. The equilibrium deformations for a given IBM Hamiltonian are found by minimizing an energy surface,  $E(\beta, \gamma)$ , obtained by its expectation value in an intrinsic state which is a condensate of  $N$  bosons,  $b_c^\dagger \propto \beta \cos \gamma d_0^\dagger + \beta \sin \gamma (d_2^\dagger + d_{-2}^\dagger)/\sqrt{2} + s^\dagger$ , that depends parametrically on  $(\beta, \gamma)$  [38, 39]. Apart from a constant,  $E(\beta, \gamma) \propto (1 + \beta^2)^{-2} \beta^2 [a - b\beta \cos 3\gamma + c\beta^2]$ , where  $a$ ,  $b$ , and  $c$  are coefficients depending on the Hamiltonian. The two extremum equations,  $\partial E/\partial\beta = \partial E/\partial\gamma = 0$ , have  $\beta = 1$  and  $\gamma = 0$  as a solution, provided  $b = 2c$ . For large  $N$ , the coefficients of  $\hat{H}_{\text{ECQF}}$  are  $b = -\omega\xi\sqrt{\frac{2}{7}}\chi/N$  and  $c = \omega[1 - \xi - \xi\chi^2/14]/N$ . Thus, in the valley of

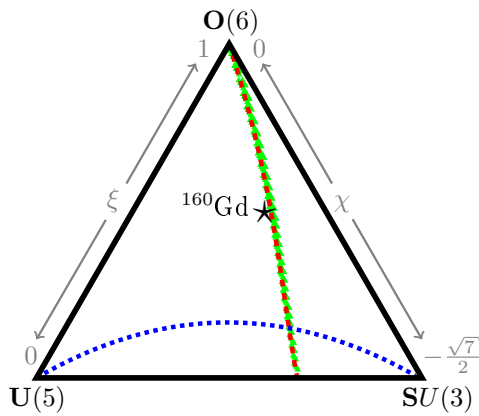


FIG. 3. (Color online) The ECQF symmetry triangle with the position of the nucleus  $^{160}\text{Gd}$  indicated by a star. The green area shows the region of low  $\Delta\sigma_{\text{gs}}$ , calculated from Eq. (3) for  $N = 60$ . The red dashed line shows the same region of approximate ground-state  $O(6)$  symmetry, as predicted by Eq. (5) for large  $N$ . The blue dotted line shows the “arc of regularity” [35].

TABLE I. Calculated  $\sigma$  fluctuations  $\Delta\sigma_L$ , Eq. (3), for rare earth nuclei in the vicinity of the identified region of approximate ground-state- $O(6)$  symmetry. Also shown are the fraction  $f_{\sigma=N}^{(L)}$  of  $O(6)$  basis states with  $\sigma = N$  contained in the  $L=0, 2, 4$  states, members of the ground band. The structure parameters  $\xi$  and  $\chi$  are taken from [30].

Nucleus	$N$	$\xi$	$\chi$	$\Delta\sigma_0$	$f_{\sigma=N}^{(0)}$	$\Delta\sigma_2$	$f_{\sigma=N}^{(2)}$	$\Delta\sigma_4$	$f_{\sigma=N}^{(4)}$
$^{156}\text{Gd}$	12	0.72	-0.86	0.46	95.3%	0.43	95.8%	0.38	96.6%
$^{158}\text{Gd}$	13	0.75	-0.80	0.35	97.2%	0.33	97.5%	0.30	97.9%
$^{160}\text{Gd}$	14	0.84	-0.53	0.19	99.1%	0.19	99.2%	0.17	99.3%
$^{162}\text{Gd}$	15	0.98	-0.53	0.41	96.0%	0.40	96.0%	0.40	96.1%
$^{160}\text{Dy}$	14	0.81	-0.49	0.44	96.2%	0.39	96.4%	0.36	96.8%
$^{162}\text{Dy}$	15	0.92	-0.31	0.07	99.9%	0.07	99.9%	0.06	99.9%
$^{164}\text{Dy}$	16	0.98	-0.26	0.13	99.6%	0.13	99.6%	0.13	99.6%
$^{164}\text{Er}$	14	0.84	-0.37	0.39	96.5%	0.37	96.7%	0.35	97.1%
$^{166}\text{Er}$	15	0.91	-0.31	0.12	99.7%	0.11	99.7%	0.10	99.7%

low  $\Delta\sigma_{\text{gs}}$  the desired condition,  $b = 2c$ , fixes  $\xi$  to be

$$\xi = \frac{1}{1 - \sqrt{\frac{1}{14}}\chi + \frac{1}{14}\chi^2}. \quad (5)$$

As seen in Fig. 3, this relation predicts the location of the region of approximate ground-state  $O(6)$  symmetry for large  $N$  very precisely. For small  $N$  its precision decreases somewhat due to finite- $N$  effects, causing a more pronounced curvature of the region close to the  $O(6)$  DS limit.

Detailed ECQF fits for energies and electromagnetic transitions of rare-earth nuclei, performed by McCutchan *et al.* [30], allow one to relate the structure of collec-

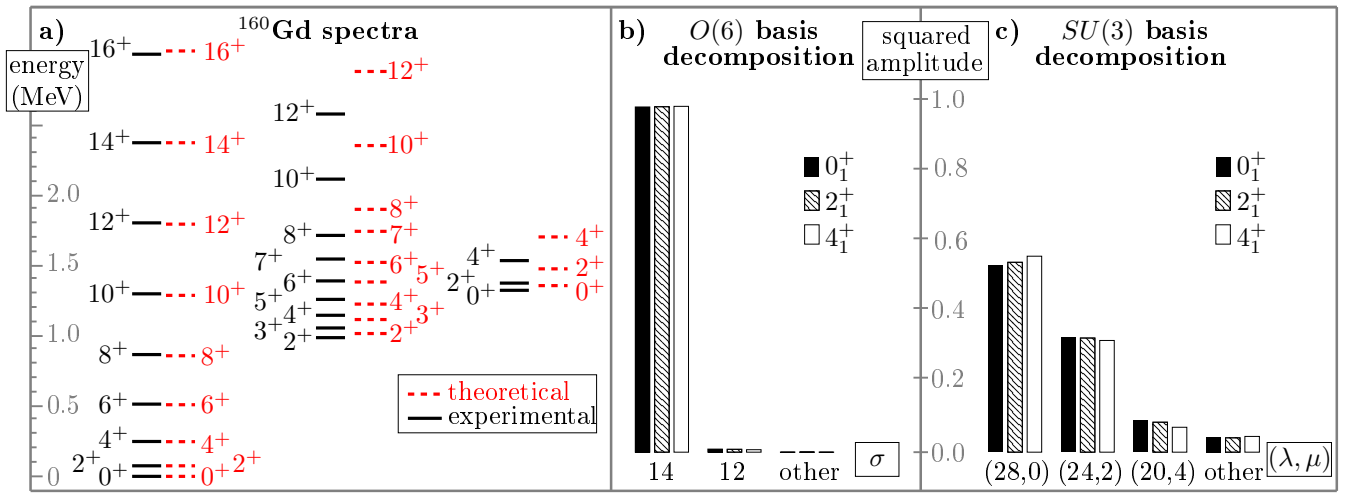


FIG. 4. (Color online) a) The experimental spectrum of  $^{160}\text{Gd}$  compared with the IBM calculation using the ECQF Hamiltonian (1) with parameters  $\xi = 0.84$  and  $\chi = -0.53$  taken from Ref. [30]. b) The  $O(6)$  decomposition in  $\sigma$  components of yrast states with  $L = 0, 2, 4$ . c) The  $SU(3)$  decomposition in  $(\lambda, \mu)$  components of the same yrast states.

tive nuclei to the parameter space of the ECQF Hamiltonian (1). Examining the extracted  $(\xi, \chi)$  parameters, one finds that several rotational nuclei in this region, such as  $^{160}\text{Gd}$ , commonly interpreted as  $SU(3)$ -like nuclei, are actually located in the valley of small  $\sigma$  fluctuations. They can be identified as candidate nuclei with approximate ground-state  $O(6)$  symmetry. The experimental spectrum of  $^{160}\text{Gd}$ , along with its ECQF description with  $\xi = 0.84$  and  $\chi = -0.53$  taken from Ref. [30], is shown in the left panel of Fig. 4. The middle and right panels show the decomposition into  $O(6)$  and  $SU(3)$  basis states, respectively, for yrast states with  $L = 0, 2, 4$ . It is evident that the  $SU(3)$  symmetry is broken, as significant contributions of basis states with different  $SU(3)$  quantum numbers  $(\lambda, \mu)$  occur. It is also clear from Fig. 4c that this mixing occurs in a coherent manner with similar patterns for the different members of the ground-state band. This is the hallmark of a QDS [18] and it results from the existence of a single intrinsic wave function for the members of this band. On the other hand, as seen in Fig. 4b, the yrast states with  $L = 0, 2, 4$  are almost entirely composed out of  $O(6)$  basis states with  $\sigma = N = 14$  which implies small fluctuations  $\Delta\sigma_\Psi$  and the preservation of  $O(6)$  symmetry in the ground-state band.

Other rare-earth nuclei with ground-state bands with approximate  $O(6)$  symmetry can be identified by the same arguments. Their structure parameters  $\xi$  and  $\chi$  can be taken from Ref. [30], from where the fluctuations  $\Delta\sigma_\Psi$  and the fractions  $f_{\sigma=N}$  of squared  $\sigma = N$  amplitude can be calculated. Nuclei with  $\Delta\sigma_{\text{gs}} < 0.5$  and  $f_{\sigma=N} > 95\%$  are listed in Table I. These quantities are also calculated for yrast states with  $L > 0$  and exhibit similar values in each nucleus. It is evident that the IBM predicts a high degree of  $O(6)$  purity in the ground-state-band, for

a large set of rotational rare-earth nuclei.

These results show that the approximate  $O(6)$  PDS does hold not only for the ground state but also for the members of the band built on top of it. Since the entire band corresponds to a single intrinsic state, the  $SU(3)$  wave-function decomposition is similar for the different members of the band and therefore the notion of  $SU(3)$  QDS applies. In addition, provided the indicated intrinsic state has  $\beta \approx 1$  and  $\gamma = 0$ , the notion of  $O(6)$  PDS applies. Thus a link is established between  $SU(3)$  QDS and  $O(6)$  PDS.

To summarize, the method of quantum-number fluctuations reveals the existence of a region of almost exact ground-state-band  $O(6)$  symmetry outside the  $O(6)$  DS limit of the IBM. The existence of a valley of small  $\sigma$  fluctuations can be understood in terms of an approximate  $O(6)$  PDS of type III. The same wave functions display coherent ( $L$ -independent) mixing of  $SU(3)$  representations and hence comply with the conditions of an  $SU(3)$  QDS. Coherent mixing of one symmetry may therefore result in the purity of a quantum number associated with partial conservation of a different, incompatible symmetry. Previously established ECQF systematics show that many rare-earth nuclei do exhibit these approximate partial  $O(6)$  and quasi  $SU(3)$  dynamical symmetries. We conclude that partial dynamical symmetries are more abundant than previously recognized, may lead to coherent mixing and quasi dynamical symmetries, and hence play a role in understanding the regular behavior of complex nuclei. This example serves to illustrate a fundamental linkage between two distinct types of intermediate symmetries, PDS and QDS, with potential implications to algebraic modeling of diverse dynamical systems.

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- [1] I. Talmi, *Simple Models of Complex Nuclei*, (Harwood Academic, 1993).
- [2] P.W. Anderson, *Science* **177**, 393 (1972).
- [3] F. Iachello, *Lie Algebras and Applications*, (Springer-Verlag, Berlin 2006).
- [4] For a review see, A. Leviatan, *Prog. Part. Nucl. Phys.* **66**, 93 (2011).
- [5] A. Leviatan, *Phys. Rev. Lett.* **77**, 818 (1996).
- [6] P. Van Isacker, *Phys. Rev. Lett.* **83**, 4269 (1999).
- [7] A. Leviatan and P. Van Isacker, *Phys. Rev. Lett.* **89**, 222501 (2002).
- [8] J. Escher and A. Leviatan, *Phys. Rev. Lett.* **84**, 1866 (2000).
- [9] D.J. Rowe and G. Rosensteel, *Phys. Rev. Lett.* **87**, 172501 (2001).
- [10] P. Van Isacker and S. Heinze, *Phys. Rev. Lett.* **100**, 052501 (2008).
- [11] J.E. García-Ramos, A. Leviatan and P. Van Isacker, *Phys. Rev. Lett.* **102**, 112502 (2009).
- [12] A. Leviatan, J.E. García-Ramos and P. Van Isacker, *Phys. Rev. C* **87**, 021302(R) (2013).
- [13] J.L. Ping and J.Q. Chen, *Ann. Phys. (N.Y.)* **255**, 75 (1997).
- [14] J. Carvalho, R. Le Blanc, M. Vassanji, D.J. Rowe and J. McGrory, *Nucl. Phys. A* **452**, 240 (1986).
- [15] C. Bahri and D.J. Rowe, *Nucl. Phys. A* **662**, 125 (2000).
- [16] P. Rochford and D.J. Rowe, *Phys. Lett. B* **210**, 5 (1988).
- [17] P.S. Turner and D.J. Rowe, *Nucl. Phys. A* **756**, 333 (2005).
- [18] D.J. Rowe, *Nucl. Phys. A* **745**, 47 (2004).
- [19] G. Rosensteel and D.J. Rowe, *Nucl. Phys. A* **759**, 92 (2005).
- [20] M. Macek, J. Dobeš and P. Cejnar, *Phys. Rev. C* **80**, 014319 (2009); **82**, 014308 (2010).
- [21] M. Macek, J. Dobeš, P. Stránský and P. Cejnar, *Phys. Rev. Lett.* **105**, 072503 (2010).
- [22] D.J. Rowe, P. Rochford and J. Repka, *J. Math. Phys. (N.Y.)* **29**, 572 (1988).
- [23] D.J. Rowe, in *Computational and Group-Theoretical Methods in Nuclear Physics*, eds. J. Escher, O. Castaños, J.G. Hirsch, S. Pittel and G. Stoitcheva, (World Scientific, Singapore, 2004), p. 165, arXiv:1106.1607 [nucl-th].
- [24] A. Bohm, Y. Néeman and A.O. Barut (Eds.), *Dynamical Groups and Spectrum Generating Algebras*, (World Scientific, 1988).
- [25] F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge University Press, Cambridge, 1987).
- [26] A. Bohr and B.R. Mottelson, *Nuclear Structure II* (Benjamin, Reading, MA, 1975).
- [27] D.D. Warner and R.F. Casten, *Phys. Rev. C* **28**, 1798 (1983).
- [28] P.O. Lipas, P. Toivonen and D.D. Warner, *Phys. Lett. B* **155**, 295 (1985).
- [29] R.F. Casten and D.D. Warner, *Prog. Part. Nucl. Phys.* **9**, 311 (1983).
- [30] E.A. McCutchan, N.V. Zamfir and R.F. Casten, *Phys. Rev. C* **69**, 064306 (2004).
- [31] A. Arima and F. Iachello, *Ann. Phys. (N.Y.)* **123**, 468 (1979).
- [32] G. Rainovski, N. Pietralla, T. Ahn, L. Coquard, C. Lister, R. Janssens, M.P. Carpenter, S. Zhu, L. Bettermann, J. Jolie, W. Rother, R.V. Jolos and V. Werner, *Phys. Lett. B* **683**, 11 (2010).
- [33] P. Cejnar and J. Jolie, *Phys. Lett. B* **420**, 241 (1998).
- [34] S. Heinze, *Eine Methode zur Lösung beliebiger bosonischer und fermionischer Vielteilchensysteme*, Dissertation, University of Cologne (2008).
- [35] Y. Alhassid and N. Whelan, *Phys. Rev. Lett.* **67**, 816 (1991).
- [36] D. Bonatsos, E.A. McCutchan and R.F. Casten, *Phys. Rev. Lett.* **104**, 022502 (2010).
- [37] A. Bohr, *Mat. Fys. Medd. Dan. Vid. Selsk.* **26**, No. 14 (1952).
- [38] J.N. Ginocchio and M.W. Kirson, *Phys. Rev. Lett.* **44**, 1744 (1980).
- [39] A.E.L. Dieperink, O. Scholten and F. Iachello, *Phys. Rev. Lett.* **44**, 1747 (1980).