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Sergey Fedoruk, Andrei Smilga

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Sergey Fedoruk [a b ‡], **Andrei Smilga** [c]

[a] *Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Moscow region, Russia*
fedoruk@theor.jinr.ru

[b] *Dept. Theor. Phys., Tomsk State Pedagogical University, Tomsk, 634061, Russia*

[c] *SUBATECH, Université de Nantes, 4 rue Alfred Kastler, BP 20722, Nantes 44307, France;*
smilga@subatech.in2p3.fr

Abstract

Using complex notation, we present new simple expressions for two pairs of complex supercharges in HKT supersymmetric sigma models. The second pair of supercharges depends on the holomorphic antisymmetric "hypercomplex structure" tensor $\mathcal{I}_{jk}$ which plays the same role for the HKT models as the complex structure tensor for the Kähler models. When the Hamiltonian and supercharges commute with the momenta conjugate to the imaginary parts of the complex coordinates, one can perform a Hamiltonian reduction. The models thus obtained represent a special class of quasicomplex sigma models introduced recently in [1].

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‡ On leave of absence from V.N. Karazin Kharkov National University, Ukraine.

1 Introduction

Supersymmetric quantum mechanical (SQM) models describing the motion of a supersymmetric particle on a curved manifold have been studied since [2]. Most of these problems represent a reformulation of classical problems of differential geometry. In particular, the model analyzed in [2] boils down to the well-known de Rham complex.

The powerful supersymmetry formalism allows one to reproduce known mathematical results in a simple way. In this regard, one can mention the famous Atiyah-Singer theorem [3]. A pure mathematical proof of this theorem is rather complicated. On the other hand, its supersymmetric proof using the functional integral formalism [4] (see also [5, 6]) is transparent and beautiful.

But supersymmetry makes it also possible to derive *new* results. In particular, it allows to construct new differential geometry structures not studied before by mathematicians. For example, the SQM model studied in [2] and involving an extra potential is called now "Witten deformation of the de Rham complex". There are other deformations of the classical de Rham and Dolbeault complexes involving torsions [7, 8]. The HKT models (the subject of the present paper) were first introduced by physicists in the supersymmetric sigma model framework [9] (see also earlier papers [10, 11, 12, 13] where some elements of the HKT structure were displayed) and only then were described in pure mathematical terms [14, 15]. Less known CKT and OKT models [16, 17, 18] are still awaiting their appreciation by mathematicians. The same concerns the recently discovered quasicomplex sigma models.

To find a way in this multitude of models, one needs road maps. We noticed in [19] that *all* these models can be obtained from the trivial flat Dolbeault model with

$$Q = \psi_a \pi_a, \qquad \bar{Q} = \bar{\psi}_a \bar{\pi}_a, \qquad H = \bar{\pi}_a \pi_a \tag{1.1}$$

by two operations: *(i)* similarity transformation of complex supercharges and *(ii)* Hamiltonian reduction. In particular, a similarity transformation

$$Q \ \to \ e^R Q e^{-R}, \qquad \bar{Q} \ \to \ e^{-R^\dagger} \bar{Q} e^{R^\dagger} \tag{1.2}$$

with $R = \omega_{ab}\psi_a\bar{\psi}_b$ applied to (1.1) gives a model describing a nontrivial Dolbeault complex. If the metric

$$h_{m\bar{n}} \ = \ \left(e^{-\omega} e^{-\omega^\dagger}\right)_{m\bar{n}} \tag{1.3}$$

thus obtained does not depend on imaginary parts of the complex coordinates z^m, the momenta $\pi_m - \bar{\pi}_m$ commute with the Hamiltonian and one can perform a Hamiltonian reduction giving a model with half as much bosonic degrees of freedom $\{\mathrm{Re}(z^m)\}$ [1]. If the Hermitian metric (1.3) involves an imaginary part,

$$h_{m\bar{n}} \ = \ \frac{1}{2}\left(g_{(m\bar{n})} + ib_{[m\bar{n}]}\right) \,, \tag{1.4}$$

we obtain a quasicomplex model [1] (the origin of the factor 1/2 in (1.4) will be clarified later). If $b_{[m\bar{n}]} = 0$, we obtain a usual de Rham model of [2].

[1] The Hamiltonian can, of course, commute with any number of momenta. The corresponding Hamiltonian reductions give different models some of which were discussed in [19]. In this paper, we will discuss only the Hamiltonian reduction with respect to all imaginary parts of z^m.

Both Dolbeault and de Rham models can have extended supersymmetries. The de Rham model with an extra pair of supercharges can be formulated for Kähler even-dimensional manifolds [20, 21, 22]. Mathematicians know this model as the Kähler – de Rham complex. There are also $\mathcal{N} = 8$ supersymmetric (i.e. including 8 different real supercharges) de Rham models with 3 extra pairs of supercharges and defined on hyper-Kähler manifolds. A Dolbeault model with an extra pair of supercharges is called an HKT model [2]. If its metric does not depend on $\mathrm{Im}(z^m)$, one can perform a Hamiltonian reduction.

Our main observation is that a model thus obtained belongs to the class of quasicomplex models representing their special type. It enjoys $\mathcal{N} = 4$ supersymmetry.

The explicit component expressions for the HKT supercharges were derived in [24]. However, they were written in terms of real coordinates. To perform the Hamiltonian reduction described above, we need first to represent them in complex form. If expressing in proper terms, the corresponding expressions turn out to be very simple [see Eq.(3.18) below]. This representation make manifest the kinship between the mathematical structure of the HKT models and the structure of Kähler – de Rham models. The latter are characterized by a presence of the closed Kähler form. The components of this form define the complex structure tensor I_{MN}. Similarly, an HKT manifold is characterized by the presence of a closed holomorphic $(2, 0)$ - form. Its components define a holomorphic tensor $\mathcal{I}_{mn}$ which may be called a *hypercomplex structure* tensor. [3]

The plan of the paper is the following. Sect. 2 represents a mathematical introduction where we *translate* many facts known to mathematicians into a language understandable to physicists. [4] In Sect. 3, after reminding how simple expressions for the supercharges can be derived in $\mathcal{N} = 2$ models (the main idea is to treat the fermions with world indices rather than the fermions with tangent space indices as basic dynamical variables), we present new nice generic expressions for the complex HKT supercharges as well as the supercharges obtained after their Hamiltonian reduction.

In Sect. 4 (the central section of the paper), we discuss the Hamiltonian reduction procedure invoking superfield formalism. A generic Dolbeault $\mathcal{N} = 2$ model is expressed via $(\mathbf{2}, \mathbf{2}, \mathbf{0})$ chiral superfields [5]. When the metric depends only on real parts of the coordinates, one can perform the Hamiltonian reduction with respect to imaginary parts. The reduced model is described in terms of $(\mathbf{1}, \mathbf{2}, \mathbf{1})$ multiplets — the imaginary parts of the coordinates are traded for auxiliary fields. Likewise, $\mathcal{N} = 4$ HKT models are described by $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplets that involve four real or two complex coordinates. After reduction, imaginary parts of the latter are traded for auxiliary fields and we are led to $(\mathbf{2}, \mathbf{4}, \mathbf{2})$. Generically, one obtained a *deformed* Kähler – de Rham complex which involves extra "quasicomplex" terms. At the superfield level, such models involve, besides the familiar Kähler potential term, a holomorphic F-term of some special form [see Eq. (4.38)].

We emphasize that this type of Hamiltonian reduction differs from the Hamiltonian re-

[2] HKT stands for *hyper-Kähler with torsion.* This name is probably a little bit misleading because these manifolds are *not* hyper-Kähler and not even Kähler, but a better one was not invented.

[3] Throughout the paper, the real tensor indices are denoted by large latin letters $M, N, \ldots$, while small latin letters $m, \bar{m}, \ldots$ are reserved for the holomorphic and antiholomorphic complex indices.

[4] Unfortunately, the papers written by mathematicians and by theorists doing mathematical physics are written in rather different languages, even when they are devoted to basically the same subject. More often than not they are mutually not understandable and translation is necessary.

[5] We follow the notation of [25] such that the numerals count the numbers of the physical bosonic, physical fermionic and auxiliary bosonic fields.

duction for hyper-Kähler manifolds [26, 27, 28] and HKT manifolds [29] studied earlier. In [26, 27, 28, 29], the reduction related models of the same type: hyper-Kählerian models to hyper-Kählerian and HKT to HKT. In our case, the reduction changes the geometry: a Dolbeault model gives after reduction a quasicomplex de Rham model and an HKT model gives a quasicomplex Kähler model.

Short conclusions are drawn in the last section.

In Appendix A, we discuss in details how *Hamiltonian* reduction is described in *Lagrangian* component formalism. In Appendix B, we present complete component Lagrangians of the original HKT theory with several interacting $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplets and of the quasicomplex Kähler – de Rham theory with several interacting $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ multiplets. In Appendix C, we give some technical details concerning establishing the correspondence between a generic HKT model admitting reduction and its reduced Kähler – de Rham quasicomplex daughter.

2 Two definitions of HKT manifolds and their equivalence.

We assume that the reader is familiar with the geometry of Kähler and hyper-Kähler manifolds. For a reader physicist, we can recommend the excellent review [30]. The basic facts are the following:

- A *Kähler manifold* is characterized by an antisymmetric complex structure tensor I_{MN}. [6] The property $I_{MN} I^{NK} = -\delta_M^K$ holds. I_{MN} is covariantly constant, $\nabla_P I_{MN} = 0$. It follows that the Kähler form $\Omega = I_{MN}\, dx^M \wedge dx^N$ is closed, $d\Omega = 0$.

- A generic complex manifold also involves an antisymmetric complex structure tensor I, but the standard covariant derivative $\nabla_P I_{MN}$ (with symmetric Christoffel symbols) does not necessarily vanish. I should satisfy, however, certain integrability conditions,

$$\nabla_{[M} I_{N]P} \;=\; I_M{}^Q I_N{}^S \nabla_{[Q} I_{S]P}\,. \tag{2.1}$$

  Eq.(2.1) amounts to the vanishing of the so called Nijenhuis tensor. [7] It is necessary to be able to define (anti)holomorphic coordinates $x^M = \{z^m, \bar z^{\bar m}\}$ with Hermitian metric, $ds^2 = 2h_{m\bar n} dz^m d\bar z^{\bar n}$ on the whole manifold. In addition, if (2.1) does not hold, nilpotent supercharges cannot be constructed.

[6] Our index policy is the following. *(i)* Capital latin letters denote the indices in $\mathcal{R}^N$. *(ii)* small latin letters are reserved for the indices of holomorphic variables. *(iii)* In most cases but not always, the indices of antiholomorphic variables are marked with a bar ($\bar z^{\bar m}$ etc) . *(iv)* By the reasons which become clear later, the holomorphic indices in Sect.4.2.3 are Greek.

[7] The Nijenhuis tensor is defined as

$$N^I_{JK} = I^M_{\ [J} \partial_M I^I_{\ K]} \;-\; I^I_{\ M} \partial_{[J} I^M_{\ K]}\,. \tag{2.2}$$

Its vanishing may be expressed as a condition

$$\partial_{[M} I_{N]}^P \;=\; I_M^Q I_N^S \partial_{[Q} I_{S]}^P\,. \tag{2.3}$$

One can observe that one can as well replace the usual derivatives in (2.3) by covariant ones. Lowering the index P then gives (2.1).

When the complex coordinates are chosen, the tensor I_M^N has the following nonzero components,

$$I_m{}^n = -I^n{}_m = -i\delta^n_m, \quad I_{\bar m}{}^{\bar n} = -I^{\bar n}{}_{\bar m} = i\delta^{\bar n}_{\bar m}\,. \tag{2.4}$$

It follows that $I_{m\bar n} = -I_{\bar n m} = -ih_{m\bar n}$.

- As was mentioned, a standard covariant derivative of I_{MN} does not generically vanish. However, for any I satisfying the conditions above, one can define an affine connection

$$\hat\Gamma^M_{NK} \;=\; \Gamma^M_{NK} + \frac{1}{2} g^{ML} C_{LNK} \tag{2.5}$$

with the torsion tensor C_{LNK} antisymmetric under $N \leftrightarrow K$ such that $\hat\nabla_P I_{MN} = 0$. If one requires for the tensor C_{LNK} to be totally antisymmetric, such connection is unique and is called *Bismut connection* [31]. Explicitly,

$$C^{(Bismut)}_{MNK}(I) \;=\; I_M{}^P I_N{}^Q I_K{}^R (\nabla_P I_{QR} + \nabla_Q I_{RP} + \nabla_R I_{PQ})\,. \tag{2.6}$$

In complex coordinates, this tensor involves only the components of the type $(2,1)$ and $(1,2)$. The explicit expressions are [5]

$$\begin{aligned} C_{mn\bar p} = C_{n\bar p m} &= C_{\bar p mn} = \; \partial_n h_{m\bar p} - \partial_m h_{n\bar p}, \\ C_{\bar m\bar n p} = C_{\bar n p\bar m} &= C_{p\bar m\bar n} = \; \partial_{\bar n} h_{\bar m p} - \partial_{\bar m} h_{\bar n p}\,. \end{aligned} \tag{2.7}$$

- A *hyper-Kähler manifold* has three different antisymmetric covariantly constant complex structures I, J, K satisfying the quaternion algebra

$$I^2 = J^2 = K^2 = -1, \qquad IJ = K, \quad JK = I, \quad KI = J\,. \tag{2.8}$$

- Finally, we define a *hypercomplex manifold* as a manifold with three integrable quaternionic complex structures whose standard covariant derivatives do not necessarily vanish. The real dimension of a hypercomplex manifold is an integer multiple of 4 — the same as for the hyper-Kähler manifolds.

We go over now to the HKT manifolds. There are two equivalent definitions:

Definition 1. An HKT manifold is a hypercomplex manifold where the complex structures satisfy an additional constraint: they are covariantly constant with *one and the same* torsionful Bismut affine connection,

$$C_{MNK}(I) \;=\; C_{MNK}(J) \;=\; C_{MNK}(K)\,. \tag{2.9}$$

Definition 2. An HKT manifold is a hypercomplex manifold where the $(2,0)$ - form

$$\omega = \; \Omega_J + i\Omega_K = \; (J+iK)_{MN}\, dx^M \wedge dx^N \tag{2.10}$$

(we will shortly see that it is holomorphic with respect to I) is closed,

$$\partial_I \omega = 0$$

.

We will give a proof here for the half of the equivalence theorem (see e.g. [15] for another half). Taking (2.9) as a basic definition (suggested originally in [9]), we construct the closed holomorphic $(2,0)$ - form. The existence of such form was first proven in [32]. We follow here much more user-physicist-friendly [14].

As a first step, we introduce two operators associated with the complex structure I and acting on $n-$ forms. The operator ι is defined according to

$$\begin{aligned} \text{if} \qquad \omega &= \omega_{M_1 \ldots M_n} dx^{M_1} \wedge \cdots \wedge dx^{M_n} , \\ \text{then} \qquad \iota\omega &= n\, \omega_{N[M_2 \ldots M_n} (I)^N_{\ M_1]} dx^{M_1} \wedge \cdots \wedge dx^{M_n} . \end{aligned} \tag{2.11}$$

For a form $\omega_{p,q}$ with p holomorphic and q antiholomorphic indices,

$$\iota\omega_{p,q} = i(p-q)\omega_{p,q} . \tag{2.12}$$

Another operator $\omega \to I\omega$ is defined as

$$I\omega = I_{M_1}^{N_1} \ldots I_{M_n}^{N_n}\, \omega_{N_1 \ldots N_n}\, dx^{M_1} \wedge \cdots \wedge dx^{M_n} . \tag{2.13}$$

When acting on the form of the type (p,q), it multiplies ω by the factor i^{q-p}.

Finally, on top of the usual exterior derivative d, we introduce the operator

$$d_I = [d, \iota] .$$

Representing d as the sum of the holomorphic and antiholomorphic (with respect to I) exterior derivatives, $d = \partial_I + \bar{\partial}_I$ and using (2.12), we easily derive $d_I = i(\bar{\partial}_I - \partial_I)$ (and hence $\partial_I = (d + i d_I)/2$). For an integrable I, complex coordinates can be chosen such the complex structure matrix (2.4) is constant. In this case, we can write a simple explicit expression for d_I,

$$d_I\omega = I_M^{\ S}\, \partial_S \omega_{N_1 \ldots N_n}\, dx^M \wedge dx^{N_1} \wedge \cdots \wedge dx^{N_n} . \tag{2.14}$$

We prove now some simple lemmas.

Proposition 1.

$$d_I\omega = (-1)^n I d(I\omega) , \tag{2.15}$$

where n is the order of the form.

Proof: Choose the complex coordinates. Consider the R.H.S. of (2.15) and use the complex expression (2.4) for I. The components $I_M^{\ N}$ are thus constant and the partial derivatives do not act upon them. The form $d(I\omega)$ has the order $n+1$ and, according to (2.13), the expression $Id(I\omega)$ has altogether $(n+1)+n = 2n+1$ factors of I. This involves n pairs giving $I^2 = -1$ [this compensates the factor $(-1)^n$] and we are left with just one unpaired factor. We obtain the expression (2.14). In contrast to (2.14), the R.H.S. of (2.15) has a tensorial form and is valid with *any* choice of coordinates.

Proposition 2. The form (2.10) has the type $(2,0)$ with respect to I.

Proof: Indeed, using the definition (2.11) and the properties (2.8), it is easy to derive $\iota\omega = 2i\omega$.

Proposition 3. For any complex manifold,

$$d_I \Omega_I \;=\; \frac{1}{3} C_{MPQ}\, dx^M \wedge dx^P \wedge dx^Q \,. \tag{2.16}$$

Proof: Choosing complex coordinates and bearing in mind (2.4), (2.14) and (2.7), we derive

$$d_I \Omega_I = C_{mp\bar{q}} dz^m \wedge dz^p \wedge d\bar{z}^{\bar{q}} + C_{\bar{m}\bar{p}q} d\bar{z}^{\bar{m}} \wedge d\bar{z}^{\bar{p}} \wedge dz^q \,,$$

which coincides with (2.16).

Corollary: For the HKT manifolds where the Bismut torsions for I, J, K coincide,

$$d_I \Omega_I = \; d_J \Omega_J = \; d_K \Omega_K \,. \tag{2.17}$$

Proposition 4. Let I, J, K be quaternion complex structures. Then

$$\begin{gathered} I\Omega_I = \Omega_I \,, \qquad J\Omega_J = \Omega_J \,, \qquad K\Omega_K = \Omega_K \,, \\ J\Omega_I = K\Omega_I = -\Omega_I \,, \qquad I\Omega_J = K\Omega_J = -\Omega_J \,, \qquad I\Omega_K = J\Omega_K = -\Omega_K \,. \end{gathered} \tag{2.18}$$

Proof: Let us prove the relation $J\Omega_I = -\Omega_I$. By definition,

$$J\Omega_I \;=\; J_M^{\ P} J_S^{\ Q} I_{PQ}\, dx^M \wedge dx^S \,.$$

On the other hand,

$$J_M^{\ P} J_S^{\ Q} I_{PQ} \;=\; -K_{MQ} J_S^{\ Q} = -I_{MS} \,.$$

Other relations are proved similarly.

Remark. The condition (2.17) can be rewritten bearing in mind (2.15) and the first line in (2.18) as

$$I d\Omega_I = J d\Omega_J = K d\Omega_K \,. \tag{2.19}$$

We are ready now to prove the main theorem

Theorem 1.

$$\partial_I (\Omega_J + i\Omega_K) \;=\; 0 \,. \tag{2.20}$$

Proof. The real and imaginary parts of (2.20) give a kind of Cauchy-Riemann conditions

$$d\Omega_J - d_I \Omega_K \;=\; 0, \qquad d\Omega_K + d_I \Omega_J = 0 \,. \tag{2.21}$$

Consider the first relation. We obtain

$$d_I \Omega_K \stackrel{1}{=} I d(I\Omega_K) \stackrel{4}{=} -I d\Omega_K = -JK d\Omega_K \stackrel{remark}{=} -J^2 d\Omega_J = d\Omega_J \,.$$

The number "1" above the equality sign means *in virtue of the Proposition 1*, etc.

The relation $d\Omega_K + d_I \Omega_J = 0$ is proved similarly.

$\square$

3 Supercharges and reduced supercharges.

3.1 De Rham, Kähler - de Rham, Dolbeault, and quasicomplex systems.

The classical supercharges of the best known de Rham SQM sigma model are usually presented in the form

$$\begin{aligned} Q &= \psi^M \left(P_M - i\Omega_{M,AB}\psi_A\bar\psi_B\right), \\ \bar Q &= \bar\psi^M \left(P_M - i\Omega_{M,AB}\bar\psi_A\psi_B\right), \end{aligned} \tag{3.1}$$

where A, B are the tangent space indices, $\psi_A = e_{AM}\psi^M$, $g_{MN} = e_{AM}e_{AN}$, and

$$\Omega_{M,AB} = e_{AN}\left(\partial_M e^N_B + \Gamma^N_{MT}e^T_B\right) \tag{3.2}$$

are spin connections. The "flat" fermion variables ψ_A, $\bar\psi_A$ constitute, together with x^M, P_M, the orthogonal canonically conjugated pairs.

For our purposes, it is more convenient to express the supercharges in terms of fermionic variables carrying world indices. The commutation relations are in this case more complicated,

$$\begin{aligned} \left\{x^M, \Pi_N\right\}_{\text{P.B.}} &= \delta^M_N\,, \qquad \left\{\psi^M, \bar\psi^N\right\}_{\text{P.B.}} = -ig^{MN}, \\ \left\{\Pi_M, \psi^N\right\}_{\text{P.B.}} &= -\tfrac12\,\partial_M g^{NQ}\psi_Q, \qquad \left\{\Pi_M, \bar\psi^N\right\}_{\text{P.B.}} = -\tfrac12\,\partial_M g^{NQ}\bar\psi_Q. \end{aligned} \tag{3.3}$$

($\{\}_{P.B.}$ stands for a Poisson bracket). On the other hand, the expressions for the supercharges become much simpler [1],

$$\begin{aligned} Q &= \psi^M \left(\Pi_M - \tfrac{i}{2}\,\partial_M g_{NP}\,\psi^N\bar\psi^P\right), \\ \bar Q &= \bar\psi^M \left(\Pi_M + \tfrac{i}{2}\,\partial_M g_{NP}\,\psi^P\bar\psi^N\right), \end{aligned} \tag{3.4}$$

Note that the momenta P_M and Π_M are *not* the same. P_M is the variation of the Lagrangian over $\dot x^M$ while keeping ψ_A and $\bar\psi_A$ fixed. And Π_M is the variation of the Lagrangian over $\dot x^M$ while keeping ψ^M and $\bar\psi^M$ fixed. These two canonical momenta are related as [8]

$$P_M = \Pi_M + \frac{i}{2}\Big[(\partial_M e_{AP})e_{AQ} - (\partial_M e_{AQ})e_{AP}\Big]\psi^P\bar\psi^Q\,. \tag{3.5}$$

The *covariant* quantum supercharges that act on the wave functions normalized with the measure $d\mu = \sqrt{\det(g)}\,d^Nx$ have the same functional form with the operators $\Pi_M = -i\partial/\partial x^M$ and $\bar\psi^M = g^{MN}\partial/\partial\psi^N$.

For Kähler manifolds, the de Rham complex can be extended to involve an extra pair of supercharges. Being expressed in the same terms as in (3.4), they acquire a very simple form [19],

$$\begin{aligned} R &= \psi^N I_N^M \left(\Pi_M - \tfrac{i}{2}\,\partial_M g_{NP}\,\psi^N\bar\psi^P\right), \\ \bar R &= \bar\psi^N I_N^M \left(\Pi_M + \tfrac{i}{2}\,\partial_M g_{NP}\,\psi^P\bar\psi^N\right), \end{aligned} \tag{3.6}$$

Similar simple expressions can be derived for the supercharges of the Dolbeault complex,

$$\begin{aligned} Q &= \sqrt2\,\psi^m \left(\Pi_m - \tfrac{i}{2}\,\partial_m h_{n\bar p}\,\psi^n\bar\psi^{\bar p}\right), \\ \bar Q &= \sqrt2\,\bar\psi^{\bar m} \left(\bar\Pi_{\bar m} + \tfrac{i}{2}\,\partial_{\bar m} h_{p\bar n}\,\psi^p\bar\psi^{\bar n}\right). \end{aligned} \tag{3.7}$$

When $h_{m\bar{n}}$ does not depend on $\mathrm{Im}(z^p)$, one can perform a Hamiltonian reduction with identification $\Pi_m \equiv \bar{\Pi}_{\bar{m}} \to \Pi_M/2$. [8] If $h_{m\bar{n}}$ are real, we obtain the de Rham supercharges (3.4). [9] For a generic Hermitian metric (1.4), we obtain the supercharges of a quasicomplex model,

$$\begin{aligned} Q &= \psi^M \left[\Pi_M - \tfrac{i}{2}\partial_M \left(g_{(NP)} + i b_{[NP]}\right)\psi^N \bar{\psi}^P\right], \\ \bar{Q} &= \bar{\psi}^M \left[\Pi_M + \tfrac{i}{2}\partial_M \left(g_{(NP)} - i b_{[NP]}\right)\psi^P \bar{\psi}^N\right]. \end{aligned} \tag{3.8}$$

3.2 HKT supercharges

The expressions for four real supercharges in an HKT model were derived in [24]. They are

$$Q = \psi^M \Big(P_M - \tfrac{i}{2}\Omega_{M,AB}\psi^A\psi^B + \tfrac{i}{12} C_{MNP}\,\psi^N\psi^P\Big), \tag{3.9}$$

$$Q^{a=1,2,3} = \psi^Q (I^a)_Q{}^M \Big(P_M - \tfrac{i}{2}\Omega_{M,AB}\psi^A\psi^B - \tfrac{i}{4} C_{MNP}\,\psi^N\chi^P\Big), \tag{3.10}$$

where Ψ^M are here *real* fermions with $\{\Psi^M, \Psi^N\}_{\mathrm{P.B.}} = -i\delta^{MN}$ and $I^a = \{I, J, K\}$.

We choose now complex coordinates $x^M = \{z^m, \bar{z}^{\bar{m}}\}$ and construct the complex combinations

$$S = \frac{Q + iQ^1}{2}, \quad \bar{S} = \frac{Q - iQ^1}{2}, \qquad R = \frac{Q^2 + iQ^3}{2}, \quad \bar{R} = \frac{Q^2 - iQ^3}{2}. \tag{3.11}$$

A short calculation gives

$$\begin{aligned} S^{\text{HKT}} &= \sqrt{2}\,\psi^m \left[P_m - i\Omega_{m,k\bar{l}}\,\psi^k\bar{\psi}^{\bar{l}}\right], \\ \bar{S}^{\text{HKT}} &= \sqrt{2}\,\bar{\psi}^{\bar{m}} \left[\bar{P}_{\bar{m}} - i\bar{\Omega}_{\bar{m},k\bar{l}}\,\psi^k\bar{\psi}^{\bar{l}}\right], \end{aligned} \tag{3.12}$$

$$\begin{aligned} R^{\text{HKT}} &= \sqrt{2}\,\psi^n \mathcal{I}_n{}^{\bar{m}} \left[\bar{P}_{\bar{m}} - i\left(\bar{\Omega}_{\bar{m},k\bar{l}} + \tfrac{1}{2} C_{\bar{m},k\bar{l}}\right)\psi^k\bar{\psi}^{\bar{l}}\right], \\ \bar{R}^{\text{HKT}} &= \sqrt{2}\,\bar{\psi}^{\bar{n}} \mathcal{I}_{\bar{n}}{}^{m} \left[P_m - i\left(\Omega_{m,k\bar{l}} + \tfrac{1}{2} C_{mk\bar{l}}\right)\psi^k\bar{\psi}^{\bar{l}}\right], \end{aligned} \tag{3.13}$$

where $\Omega_{m,k\bar{l}} = \Omega_{m,a\bar{b}} e^a_k e^{\bar{b}}_{\bar{l}}$ and $\mathcal{I} = J + iK$.

It is noteworthy that in the expressions for S and $\bar{S}$, the torsions $C_{mk\bar{l}}$, $C_{\bar{m}k\bar{l}}$ cancelled such that $S, \bar{S}$ represent usual Dolbeault supercharges (cf. (3.15) of Ref.[5]). The torsions enter, however in R and $\bar{R}$. For hyper-Kähler manifolds, there are no torsions and the expressions (3.13) simplify.

Substituting the explicit expressions of Ω and C via vielbeins,

$$\Omega_{m,k\bar{l}} = e^{\bar{a}}_{\bar{l}}\partial_{[m} e^a_{k]} - e^a_{(m}\partial_{k)} e^{\bar{a}}_{\bar{l}}, \qquad \bar{\Omega}_{\bar{m},k\bar{l}} = -e^a_k \bar{\partial}_{[\bar{m}} e^{\bar{a}}_{\bar{l}]} + e^{\bar{a}}_{(\bar{m}}\partial_{\bar{l})} e^a_k, \tag{3.14}$$

[8] This implies the convention

$$z = x + iy, \quad \bar{z} = x - iy, \qquad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

for each complex coordinate.

[9] When deriving this, we have to take into account (1.4) and to bear in mind that the canonical de Rham fermions ψ^M carrying the world indices and satisfying (3.3) have an additional factor $1/\sqrt{2}$ compared to ψ^m.

$$\tfrac{1}{2}\, C_{mk\bar{l}} = -e^{\bar{a}}_{\bar{l}} \partial_{[m} e^{a}_{k]} + e^{a}_{[m} \partial_{k]} e^{\bar{a}}_{\bar{l}}\,, \qquad \tfrac{1}{2}\, \bar{C}_{\bar{m}\bar{l}k} = -e^{a}_{k} \bar{\partial}_{[\bar{m}} e^{\bar{a}}_{\bar{l}]} + e^{\bar{a}}_{[\bar{m}} \partial_{\bar{l}]} e^{a}_{k}\,, \tag{3.15}$$

we derive for the supercharges

$$\begin{aligned} S^{\text{HKT}} &= \sqrt{2}\, \psi^m \left[P_m - i \left(e^{\bar{a}}_{\bar{l}} \partial_m e^{a}_{k} \right) \psi^k \bar{\psi}^{\bar{l}} \right], \\ \bar{S}^{\text{HKT}} &= \sqrt{2}\, \bar{\psi}^{\bar{m}} \left[\bar{P}_{\bar{m}} + i \left(e^{a}_{k} \bar{\partial}_{\bar{m}} e^{\bar{a}}_{\bar{l}} \right) \psi^k \bar{\psi}^{\bar{l}} \right], \end{aligned} \tag{3.16}$$

$$\begin{aligned} R^{\text{HKT}} &= \sqrt{2} \psi^n \mathcal{I}_n{}^{\bar{m}} \left[\bar{P}_{\bar{m}} - i \left(e^{\bar{a}}_{\bar{l}} \bar{\partial}_{\bar{m}} e^{a}_{k} \right) \psi^k \bar{\psi}^{\bar{l}} \right], \\ \bar{R}^{\text{HKT}} &= \sqrt{2} \bar{\psi}^{\bar{n}} \mathcal{I}_{\bar{n}}{}^{m} \left[P_m + i \left(e^{a}_{k} \partial_m e^{\bar{a}}_{\bar{l}} \right) \psi^k \bar{\psi}^{\bar{l}} \right]. \end{aligned} \tag{3.17}$$

At the last step, we go over from the momenta P_m, $\bar{P}_{\bar{m}}$ to the momenta Π_m, $\bar{\Pi}_{\bar{m}}$ (which are relevant when ψ^m and $\bar{\psi}^{\bar{m}}$ rather than ψ^a and $\bar{\psi}^{\bar{a}}$ are treated as fundamental dynamic variables) according to (3.5). The supercharges take the simple nice form

$$\begin{aligned} S^{\text{HKT}} &= \sqrt{2}\, \psi^m \left[\Pi_m - \tfrac{i}{2} \left(\partial_m h_{k\bar{l}} \right) \psi^k \bar{\psi}^{\bar{l}} \right], \\ \bar{S}^{\text{HKT}} &= \sqrt{2}\, \bar{\psi}^{\bar{m}} \left[\bar{\Pi}_{\bar{m}} + \tfrac{i}{2} \left(\bar{\partial}_{\bar{m}} h_{k\bar{l}} \right) \psi^k \bar{\psi}^{\bar{l}} \right], \end{aligned} \tag{3.18}$$

$$\begin{aligned} R^{\text{HKT}} &= \sqrt{2}\, \psi^n \mathcal{I}_n{}^{\bar{m}} \left[\bar{\Pi}_{\bar{m}} - \tfrac{i}{2} \left(\bar{\partial}_{\bar{m}} h_{k\bar{l}} \right) \psi^k \bar{\psi}^{\bar{l}} \right], \\ \bar{R}^{\text{HKT}} &= \sqrt{2}\, \bar{\psi}^{\bar{n}} \mathcal{I}_{\bar{n}}{}^{m} \left[\Pi_m + \tfrac{i}{2} \left(\partial_m h_{k\bar{l}} \right) \psi^k \bar{\psi}^{\bar{l}} \right]. \end{aligned} \tag{3.19}$$

We observe a remarkable similarity with (3.4), (3.6). For an HKT manifold, the matrix $\mathcal{I}_n{}^{\bar{m}}$ plays the same role as the usual complex structure for the Kähler - de Rham complex. $\mathcal{I}$ can thus be called the matrix of *hypercomplex structure*. The form $\mathcal{I}_{mn}\, dz^m \wedge dz^n$ is closed, as dictated by (2.20).

When $h_{m\bar{n}}$ does not depend on the imaginary coordinate parts, one can perform the Hamiltonian reduction. As an HKT manifold is a complex manifold of a special kind, we obtain after reduction a quasicomplex model of a special kind. The reduced supercharges are

$$\begin{aligned} S^{\text{quasi}} &= \psi^M \left[\Pi_M - \tfrac{i}{2}\, \partial_M \left(g_{KL} + i b_{KL} \right) \psi^K \bar{\psi}^L \right], \\ \bar{S}^{\text{quasi}} &= \bar{\psi}^M \left[\Pi_M + \tfrac{i}{2}\, \partial_M \left(g_{KL} + i b_{KL} \right) \psi^K \bar{\psi}^L \right], \end{aligned} \tag{3.20}$$

$$\begin{aligned} R^{\text{quasi}} &= \psi^N \mathcal{I}_N{}^M \left[\Pi_M - \tfrac{i}{2}\, \partial_M \left(g_{KL} + i b_{KL} \right) \psi^K \bar{\psi}^L \right], \\ \bar{R}^{\text{quasi}} &= \bar{\psi}^N \mathcal{I}_N{}^M \left[\Pi_M + \tfrac{i}{2}\, \partial_M \left(g_{KL} + i b_{KL} \right) \psi^K \bar{\psi}^L \right] \end{aligned} \tag{3.21}$$

When the imaginary part of the metric b_{KL} vanish, the supercharges (3.20), (3.21) boil down to the Kähler supercharges (3.4), (3.6). When it does not, we are dealing with the *Kähler quasicomplex* model to be discussed in more details in the next section.

4 Hamiltonian reduction and superfields.

Hamiltonians of supersymmetric systems are expressed in components, and Hamiltonian reduction is usually described in components too — see the component expressions for the reduced supercharges (3.8), (3.20), (3.21) in the previous section. But it is interesting and instructive to see what does it correspond to in Lagrangian superfield formulation.

4.1 Dolbeault → quasicomplex de Rham.

The Dolbeault complex is described by a set of chiral complex $(\mathbf{2},\ \mathbf{2},\ \mathbf{0})$ superfields Z^m [5]. They are expressed into components as

$$Z^m \;=\; z^m + \sqrt{2}\,\theta\psi^m - i\theta\bar{\theta}\dot{z}^m \tag{4.1}$$

The corresponding supersymmetry transformations are

$$\delta z^m = -\sqrt{2}\,\epsilon\psi^m\,, \qquad \delta\psi^m = i\sqrt{2}\,\bar{\epsilon}\dot{z}^m\,, \tag{4.2}$$

$$\delta\bar{z}^m = \sqrt{2}\,\bar{\epsilon}\bar{\psi}^m\,, \qquad \delta\bar{\psi}^m = -i\sqrt{2}\,\epsilon\dot{\bar{z}}^m\,. \tag{4.3}$$

We set now $\psi^m = \sqrt{2}\,\chi^m$, $z^m = x^m + iy^m$ to obtain

$$\delta x^m = -\epsilon\chi^m + \bar{\epsilon}\bar{\chi}^m\,, \qquad \delta y^m = i(\epsilon\chi^m + \bar{\epsilon}\bar{\chi}^m)\,, \tag{4.4}$$

$$\delta\chi^m = \bar{\epsilon}(i\dot{x}^m - \dot{y}^m)\,, \qquad \delta\bar{\chi}^m = -\epsilon(i\dot{x}^m + \dot{y}^m)\,. \tag{4.5}$$

and observe that (4.2) *coincides* with the transformation law for a $(\mathbf{1},\ \mathbf{2},\ \mathbf{1})$ real superfield

$$X^m \equiv\; X^M =\; x^M + \theta\chi^M + \bar{\chi}^M\bar{\theta} + B^M\theta\bar{\theta}, \tag{4.6}$$

if identifying $\dot{y}^m = \mathrm{Im}(\dot{z}^m) \equiv B^M$. [10]

The Lagrangian of the pure Dolbeault sigma model (without gauge field) is expressed via chiral superfields as

$$L \;=\; -\tfrac{1}{4}\int d\bar{\theta}d\theta\, h_{m\bar{n}}(Z,\bar{Z})DZ^m\bar{D}\bar{Z}^{\bar{n}} \tag{4.7}$$

with Hermitian metric $h_{m\bar{n}}$ and

$$D_\theta = \partial_\theta - i\bar{\theta}\partial_t\,, \qquad \bar{D}_\theta = -\partial_{\bar{\theta}} + i\theta\partial_t\,. \tag{4.8}$$

If the metric does not depend on $\mathrm{Im}(z^m)$, one can perform the Hamiltonian reduction. The reduced *Lagrangian* should be expressed via the superfields X^M as

$$L^{\text{reduced}} \;=\; -\frac{1}{2}\int d\bar{\theta}d\theta\,[g_{(MN)}(X) + ib_{[MN]}(X)]\,DX^M\bar{D}X^N\,, \tag{4.9}$$

where $g/2$ and $b/2$ are real and imaginary parts of the Hermitian metric h, according to (1.4). Heuristically, (4.9) is obtained from (4.7) by substituting $Z^m, \bar{Z}^{\bar{m}} \to 2X^M$, while taking into account (1.4). When $h_{m\bar{n}}$ is real, this is the usual de Rham model. When it involves an antisymmetric imaginary part, we arrive at the quasicomplex de Rham model of Ref.[1].

The fact that the reduction of (4.7) gives (4.9) looks very natural. It can be accurately derived in the following way: *(i)* Express the Lagrangian (4.7) into components. *(ii)* It is invariant under the shifts $y^m \to y^m + c^m$ (the corollary of the fact that the Hamiltonian commutes with the corresponding canonical momentum). In other words, it does not explicitly depend on y^m, but only on $\dot{y}^m$. *(iii)* Substitute $\sqrt{2}\chi^M$ for ψ^m and B^M for $\dot{y}^m$. The result coincides with the component expansion of (4.9).

A more detailed justification of this procedure at the component level is given in Appendix A.

[10] The observation that the supertransformation laws for the multiplets with the same net number of the fermionic and bosonic components, but with a different distribution of the latter among the dynamic and auxiliary fields, coincide under such identification was made long time ago in [33, 34]. This was discussed in the Hamiltonian reduction context in [35] and in gauging approach (when the Hamiltonian commutes with $\mathrm{Im}(\Pi_m)$, one can impose the first class constraint $\mathrm{Im}(\Pi_m) = 0$ and treat the system as a gauge one) in [36].

4.2 HKT → quasicomplex Kähler.

Consider now the $\mathcal{N}=4$ supersymmetric HKT model. The Lagrangian is expressed via linear [11] $\mathcal{N}=4$ supermultiplets of the type $(\mathbf{4},\mathbf{4},\mathbf{0})$ [38, 16, 39, 40]. A $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet lives in the $\mathcal{N}=4$ superspace with the coordinates $(t,\theta^{ik'})$, $(\overline{\theta^{ik'}}) = -\epsilon_{ij}\epsilon_{k'l'}\theta^{jl'} \equiv -\theta_{k'j}$. The indices $i=1,2$ and $k'=1,2$ are doublet indices of the $\mathrm{SU}_L(2)$ and $\mathrm{SU}_R(2)$ groups respectively, which form the full automorphism group $\mathrm{SO}(4)=\mathrm{SU}_L(2)\times\mathrm{SU}_R(2)$ of the $\mathcal{N}{=}4$ superalgebra. Each multiplet carries a 4-vector or two spinor indices. Its component decomposition is

$$\mathcal{X}^{il'} = x^{il'} - \theta^{ik'}\chi^{l'}_{k'} + i\theta^{ik'}\theta_{k'k}\dot{x}^{kl'} - \tfrac{i}{3}\,\theta^{ii'}\theta_{i'k}\theta^{kk'}\dot{\chi}^{l'}_{k'} - \tfrac{1}{12}\,\theta^{kk'}\theta_{k'j}\theta^{ji'}\theta_{i'k}\,\ddot{x}^{\,kl'}, \tag{4.10}$$

and so it encompasses four real bosonic component fields $(\overline{x^{ik'}}) = -\epsilon_{ij}\epsilon_{k'l'}x^{jl'}$ and four real fermionic component fields $(\overline{\chi^{i'k}}) = -\epsilon_{i'j'}\epsilon_{kl}\chi^{j'l}$.

The set $x^{ik'}$ satisfying the pseudoreality condition can be represented as 4 real coordinates

$$x^M = \tfrac{1}{2}\,(\sigma^M)_{k'i}x^{ik'}\,, \qquad\qquad \sigma^M = (\vec{\sigma}, i) \tag{4.11}$$

or else as two complex coordinates v^m, $m=1,2$, [12]

$$x^{ik'} \;=\; \begin{pmatrix} \bar{v}^2 & \bar{v}^1 \\ v^1 & -v^2 \end{pmatrix}. \tag{4.12}$$

The same holds for the superfields $\mathcal{X}^{ik'}$.

The second representation (via two complex coordinates) is convenient when performing the Hamiltonian reduction. We may represent $v^m = x^m + iy^m$ and express the laws of supersymmetry transformations via x^m and y^m. Similar to what was the case for the $\mathcal{N}=2$ superfields, one can be convinced that these laws coincide with the supersymmetry transformations for the $(\mathbf{2},\mathbf{4},\mathbf{2})$ multiplet if identifying $\dot{y}^m$ with the auxiliary fields B^M (see [41] for the discussion of the reduction $(\mathbf{4},\mathbf{4},\mathbf{0}) \to (\mathbf{2},\mathbf{4},\mathbf{2})$ in superfield language using gauging procedure). Thus, to perform the Hamiltonian reduction using the Lagrangian language, one should only substitute $\dot{y}^m \to B^M$ in the component expression for the Lagrangian.

A wide class of HKT models are described by the superfield Lagrangian involving n $(\mathbf{4},\mathbf{4},\mathbf{0})$ linear multiplets,

$$L = \int d^4\theta\, \mathcal{L}(\mathcal{X}_\alpha)\,, \tag{4.13}$$

where $\alpha = 1,\ldots,n$ is the flavor index.

4.2.1 4-dimensional model.

Consider as the simplest example the model with only one multiplet, $n=1$.

The simplest HKT metric is a conformally flat metric in 4 dimensions,

$$ds^2 = G(x)\,dx^M dx^M\,. \tag{4.14}$$

[11] There are also models expressed via nonlinear multiplets [37], which we will not discuss here.

[12] We are changing notation here reserving the symbol z for the complex coordinates of the *reduced* model, see Eq.(4.29) below.

The complex structures can be chosen as

$$I_M^{\ N} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; J_M^{\ N} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \; K_M^{\ N} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \quad (4.15)$$

These are self-dual matrices expressible via 't Hooft symbols. The characteristic for a HKT manifold closed holomorphic form is

$$\omega = \Omega_J + i\Omega_K = 2\, G(x)\, dv^1 \wedge dv^2 \,, \qquad (4.16)$$

i.e. $\mathcal{I}_{mn} = G(x)\,\epsilon_{mn}$. The complex supercharges (3.18) with this hypercomplex structure matrix were written down in [24, 19]. When G depends only on $\mathrm{Re}(v^m)$, one can perform a Hamiltonian reduction. After that, we are left with only one complex coordinate $z = \mathrm{Re}(v^1) + i\mathrm{Re}(v^2)$, the complex metric involves only one component and is real. We obtain the usual $\mathcal{N} = 4$ Kähler model on a manifold of real dimension 2.

Note that in Ref.[1] a certain nontrivial quasicomplex 2-dimensional model was constructed and studied. It was observed that the spectrum of this model involves degenerate quartets and enjoys $\mathcal{N} = 4$ supersymmetry. The corresponding metric cannot be obtained, however, by a Hamiltonian reduction of an HKT metric and the observed extended supersymmetry has a different origin.

4.2.2 $4n$–dimensional model.

When $n > 1$, the situation becomes more complicated and more interesting. The 8–dimensional model described by two linear $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplets was studied in details in [18]. We consider here a model with an arbitrary number n of such multiplets. Anticipating a subsequent reduction, it is convenient to use the complex notation and describe each of them as the $\mathcal{N}{=}\,4$ superfield $\mathcal{V}^m_\alpha(t;\,\theta,\eta,\bar\theta,\bar\eta)$ where $\theta,\eta,\bar\theta,\bar\eta$ are odd coordinates of $\mathcal{N}{=}\,4$ superspace and $m = 1, 2$. The fields $\mathcal{V}^m$ are related to $\mathcal{X}^{ik'}$ as in (4.12).

To make things quite transparent, we can represent them in terms of $\mathcal{N}{=}\,2$ superfields. Each superfield $\mathcal{V}^m_\alpha$ is expressed via two $(\mathbf{2}, \mathbf{2}, \mathbf{0})$ complex superfields V^m_α and their conjugates. Its expansion of in $\eta, \bar\eta$ reads [18]

$$\mathcal{V}^m_\alpha = V^m_\alpha + \eta\, \epsilon^{mn} \bar D \bar V^{\bar n}_\alpha - i\eta\bar\eta\, \dot V^m_\alpha \,, \qquad \bar{\mathcal{V}}^{\bar m}_\alpha = \bar V^{\bar m}_\alpha - \bar\eta\, \epsilon^{\bar m \bar n} D V^n_\alpha + i\eta\bar\eta\, \dot{\bar V}^{\bar m}_\alpha \,, \qquad (4.17)$$

with $D \equiv D_\theta$ and $\bar D \equiv \bar D_\theta$ defined in (4.8). $V^m_\alpha(\theta,\bar\theta)$ are chiral superfields, $\bar D_\theta V^m_\alpha = 0$, $D_\theta \bar V^{\bar m}_\alpha = 0$. They have a standard component expansion

$$V^m_\alpha = v^m_\alpha + \sqrt{2}\,\theta\,\psi^m_\alpha - i\theta\bar\theta\,\dot v^m_\alpha \,, \qquad \bar V^{\bar m}_\alpha = \bar v^{\bar m}_\alpha - \sqrt{2}\,\bar\theta\,\bar\psi^{\bar m}_\alpha + i\theta\bar\theta\,\dot{\bar v}^{\bar m}_\alpha \,, \qquad (4.18)$$

The superfield action is

$$S = \tfrac14 \int dt\, d\theta d\bar\theta\, d\eta d\bar\eta\, \mathcal{L}(\mathcal{V}_\alpha, \bar{\mathcal{V}}_\alpha) = \tfrac14 \int dt\, d\theta d\bar\theta\, \left(\Delta^{\alpha\beta}_{m\bar n}\mathcal{L}\right) D V^m_\alpha \bar D \bar V^{\bar n}_\beta \,, \qquad (4.19)$$

where

$$\Delta^{\alpha\beta}_{m\bar n}\mathcal{L} \equiv \frac{\partial^2 \mathcal{L}(V,\bar V)}{\partial V^m_\alpha \partial \bar V^{\bar n}_\beta} + \epsilon_{mk}\epsilon_{\bar n \bar l} \frac{\partial^2 \mathcal{L}(V,\bar V)}{\partial \bar V^{\bar k}_\alpha \partial V^l_\beta} \,. \qquad (4.20)$$

Note that, for $\alpha=\beta$,

$$\Delta^{\alpha\alpha}_{m\bar n} = \delta_{m\bar n}\frac{\partial^2\mathcal{L}}{\partial V^k_\alpha \partial \bar V^{\bar k}_\alpha} \qquad \text{(here no summation with respect to } \alpha). \tag{4.21}$$

The R.H.S. of Eq.(4.19) expresses the action in terms of the $\mathcal{N}=2$ superfields. It is invariant under "hidden" supersymmetry transformations [37]

$$\delta_\eta V^m_\alpha = -\epsilon_\eta \epsilon^{mn}\bar D \bar V^{\bar n}_\alpha, \qquad \delta_\eta \bar V^{\bar m}_\alpha = \bar\epsilon_\eta \epsilon^{\bar m\bar n} D V^n_\alpha\,. \tag{4.22}$$

Integrating it further over $d^2\theta$, we obtain the component Lagrangian. Its bosonic part [13] reads

$$L_b \;=\; \left(\Delta^{\alpha\beta}_{m\bar n}\mathcal{L}\right)\dot v^m_\alpha \dot{\bar v}^{\bar n}_\beta\,, \tag{4.23}$$

The Lagrangian (4.23) implies the target space metric

$$ds^2 = \left(\Delta^{\alpha\beta}_{m\bar n}\mathcal{L}\right) dv^m_\alpha d\bar v^{\bar n}_\beta\,. \tag{4.24}$$

The closed holomorphic form is

$$\Omega = \tfrac{1}{2}\,\epsilon_{mk}\,\frac{\partial^2\mathcal{L}}{\partial \bar v^{\bar k}_\alpha \partial v^n_\beta}\, dv^m_\alpha \wedge dv^n_\beta\,. \tag{4.25}$$

4.2.3 The reduced model and its superfield description.

When L is real, the metric (4.24) is Hermitian, but not necessarily symmetric and real. To be able to perform the Hamiltonian reduction, we have to impose an extra constraint for the metric to be independent on $\mathrm{Im}(v^m_\alpha)$. This implies also certain constraints on $\mathcal{L}$. A generic admissible form of $\mathcal{L}$ will be written and discussed in Appendix C. Here we write a restricted Ansatz for $\mathcal{L}$ generating the metric with a constant antisymmetric under $\alpha\leftrightarrow\beta$ part,

$$\mathcal{L} \;= \mathcal{K} - \;\tfrac{i}{2}\,\mathcal{C}_{\alpha\beta}\left(\mathcal{V}^1_\alpha\bar{\mathcal{V}}^1_\beta - \mathcal{V}^2_\alpha\bar{\mathcal{V}}^2_\beta\right) \tag{4.26}$$

with a real function $\mathcal{K}$ generating the real symmetric part of the target space metric (4.24) that does not depend on $\mathrm{Im}(v^m_\alpha)$, and a real constant antisymmetric $\mathcal{C}_{\alpha\beta}=-\mathcal{C}_{\beta\alpha}$ (the coefficients are chosen for further convenience).

Consider the second term in (4.26). Its contribution to the bosonic kinetic Lagrangian reads

$$L_{\rm bos} \;=\; -2\,\mathcal{C}_{\alpha\beta}\left(\dot x^1_\alpha \dot y^1_\beta - \dot x^2_\alpha \dot y^2_\beta\right)\,. \tag{4.27}$$

Bearing in mind our recipe $\dot y^m_\alpha \to B^M_\alpha$, this gives

$$L^{\rm red}_{\rm bos} \;=\; -2\,\mathcal{C}_{\alpha\beta}\left(\dot x^1_\alpha B^1_\beta - \dot x^2_\alpha B^M_\beta\right) \tag{4.28}$$

in the reduced Lagrangian. The presence of the structure (4.28) is characteristic to quasicomplex de Rham models — see Eq.(B.8). In our case, we are dealing with an $\mathcal{N}=4$ model, a deformation of the Kähler - de Rham complex not studied before. To reveal Kählerian nature

[13]See Appendix B for the complete expression.

of the reduced model, we introduce *new* complex coordinates (made out of the real parts of v^m_α and of B^M_α),

$$z^\alpha = x^1_\alpha + i x^2_\alpha\,, \qquad A^\alpha = B^1_\alpha + i B^2_\alpha\,. \tag{4.29}$$

In this variables, the full bosonic Lagrangian of the reduced model has the form

$$L_b = h_{\alpha\bar\beta}(z,\bar z)\left(\dot z^\alpha \dot{\bar z}^{\bar\beta} + A^\alpha \bar A^{\bar\beta}\right) - \mathcal{C}_{[\alpha\beta]}\left(\dot z^\alpha A^\beta + \dot{\bar z}^{\bar\alpha}\bar A^{\bar\beta}\right)\,, \tag{4.30}$$

where

$$h_{\alpha\bar\beta} \;=\; \frac{\partial^2 \mathcal{K}}{\partial z^\alpha \partial \bar z^{\bar\beta}} \tag{4.31}$$

Besides the familiar first term with the Kähler metric, the Lagrangian (4.30) involves also an extra term involving $C_{[\alpha\beta]}$. In modifies the *full* complex metric obtained after excluding the the auxiliary fields A^α, $\bar A^{\bar\beta}$,

$$\tilde h_{\alpha\bar\beta} \;=\; h_{\alpha\bar\beta} + C_{[\alpha\delta]} h^{\bar\gamma\delta} C_{[\bar\gamma\bar\beta]}\,, \tag{4.32}$$

where $h^{\bar\alpha\beta}h_{\beta\bar\gamma} = \delta^{\bar\alpha}_{\bar\gamma}, h_{\alpha\bar\gamma}h^{\bar\gamma\beta} = \delta^\beta_\alpha$.

We will see now how this system is expressed in terms of the $(\mathbf{2},\,\mathbf{4},\,\mathbf{2})$ superfields. In contrast to the $n = 1$ case where the extra terms in (4.30) were absent, and the reduced model is a well-known Kähler – de Rham model, with the Lagrangian representing a superspace integral of the Kähler potential, when $n \geq 2$, the superfield Lagrangian is somewhat more complicated.

We consider a set of n chiral $(\mathbf{2},\,\mathbf{4},\,\mathbf{2})$ multiplets $\mathcal{Z}^\alpha(t;\,\theta,\bar\theta,\eta,\bar\eta)$ satisfying the constraints

$$\bar D_\theta\,\mathcal{Z}^\alpha = 0\,, \qquad \bar D_\eta\,\mathcal{Z}^\alpha = 0\,, \tag{4.33}$$

where D_θ, $\bar D_\theta$ are defined by (4.8) and

$$D_\eta = \partial_\eta - i\bar\eta\partial_t\,, \qquad \bar D_\eta = -\partial_{\bar\eta} + i\eta\partial_t\,. \tag{4.34}$$

It is convenient to represent $\mathcal{Z}^\alpha$ via $\mathcal{N} = 2$ superfields [5]: the usual chiral $(\mathbf{2},\,\mathbf{2},\,\mathbf{0})$ superfield $Z^\alpha(\theta,\bar\theta)$ and the superfield $\Phi^\alpha(\theta,\bar\theta)$ of the type $(\mathbf{0},\,\mathbf{2},\,\mathbf{2})$,

$$\mathcal{Z}^\alpha \;=\; Z^\alpha + \sqrt{2}\,\eta\,\Phi^\alpha - i\eta\bar\eta\,\dot Z^\alpha\,, \tag{4.35}$$

where

$$Z^\alpha = z^\alpha + \sqrt{2}\,\theta\,\phi^\alpha - i\theta\bar\theta\,\dot z^\alpha\,, \qquad \bar D_\theta\, Z^\alpha = 0\,, \tag{4.36}$$

$$\Phi^\alpha = \varphi^\alpha + \sqrt{2}\,\theta\, A^\alpha - i\theta\bar\theta\,\dot\varphi^\alpha\,, \qquad \bar D_\theta\,\Phi^\alpha = 0\,. \tag{4.37}$$

In (4.36), (4.37) the dynamical fields z^α and complex auxiliary fields A^α are bosonic whereas ϕ^α, φ^α are fermionic.

Now, the standard Kähler model is described by the action $\sim \int dt\, d\theta d\bar\theta\, d\eta d\bar\eta\, \mathcal{K}(\mathcal{Z},\bar{\mathcal{Z}})$. We note that one can add to this expression F-terms of a certain particular form, [14]

$$S = \tfrac{1}{4}\int dt\, d\theta d\bar\theta\, d\eta d\bar\eta\, \mathcal{K}(\mathcal{Z},\bar{\mathcal{Z}}) + \tfrac{1}{4}\int dt\, d\theta\, d\eta\, \mathcal{F}_{\alpha\beta}(\mathcal{Z})\mathcal{Z}^\alpha\dot{\mathcal{Z}}^\beta - \tfrac{1}{4}\int dt\, d\bar\theta\, d\bar\eta\, \bar{\mathcal{F}}_{\bar\alpha\bar\beta}(\bar{\mathcal{Z}})\bar{\mathcal{Z}}^{\bar\alpha}\dot{\bar{\mathcal{Z}}}^{\bar\beta}\,. \tag{4.38}$$

[14] This is specific for $d = 1$. In $d \geq 2$ field theories, it is absent. Probably, this is the reason why such a structure was not considered before.

The action is expressed in terms of the $\mathcal{N}=2$ superfields (4.36), (4.37) as follows,

$$S = -\frac{1}{4}\int dt\, d\bar\theta d\theta\, h_{\alpha\bar\beta}(Z,\bar Z)\left(DZ^\alpha \bar D \bar Z^{\bar\beta} - 2\Phi^\alpha \bar\Phi^{\bar\beta}\right) + \frac{1}{\sqrt{2}}\left[\int dt\, d\theta\, \mathcal{C}_{[\alpha\beta]}(Z)\,\Phi^\alpha \dot Z^\beta \ + \ \text{c.c.}\right] \tag{4.39}$$

where

$$\mathcal{C}_{\alpha\beta} = \ \mathcal{F}_{\alpha\beta} + Z^\gamma \partial_\alpha \mathcal{F}_{\gamma\beta}\,. \tag{4.40}$$

The full component expression of this Lagrangian is written in Appendix B. Its bosonic part (B.4) depends on the holomorphic antisymmetric tensor $C_{[\alpha\beta]}$. The expression (4.30) corresponds to the particular choice of constant real $\mathcal{F}_{\alpha\beta} = \mathcal{C}_{\alpha\beta}$.

Alternatively, one can express the Lagrangian of this deformed Kähler model via an even number of real (**1**, **2**, **1**) multiplets [obviously, (**2**, **4**, **2**) = (**2**, **2**, **0**) + (**0**, **2**, **2**) = (**1**, **2**, **1**) + (**1**, **2**, **1**)]. The model represents then a particular case of the quasicomplex de Rham model (4.9), with the metric $g_{(MN)}$ and the real antisymmetric tensor $b_{[MN]}$ having a particular form depending on the Hermitian $h_{\alpha\bar\beta}$ and the holomorphic $C_{[\alpha\beta]}$ in Eq.(4.30).

5 Conclusions.

We list again here the most essential original observations made in this paper.

1. We derived the new simple representation (3.18) for the HKT supercharges. In contrast to [24], the supercharges are expressed via complex coordinates and the fermion variables with world (rather than the tangent space) indices. The second pair of the supercharges involves the holomorphic matrix $\mathcal{I}_{mn}$ of *hypercomplex structure.*

2. We presented the new quasicomplex Kähler – de Rham model (4.38) where, in addition to the standard Kähler structure, the Lagrangian involves extra F-terms of a certain particular form.

3. We have shown that the models of this kind are obtained after a Hamiltonian reduction of HKT models. We discussed and justified the known recipe, according to which the canonical velocities corresponding to the variables subject to reduction in the original Lagrangian should be replaced by the auxiliary fields, $\dot y^m \to B^M$. In the beginning of Sect. 4, we showed how the supertransformation laws of the original multiplet and the reduced multiplet match. In Appendix A, we explored how the Hamiltonian reduction works at the Lagrangian component level for a wide class of systems (not necessarily supersymmetric.

   It is interesting to see how this procedure works for CKT and OKT models. What kind of models are obtained as a result of their Hamiltonian reduction ? This question is under study now.

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Appendix A: Hamiltonian reduction in component Lagrangian language.

As was discussed in the main text, the reduced component Lagrangian is obtained from the original Lagrangian by trading time derivatives of the coordinates subject to reduction (in our case, time derivatives of the imaginary coordinates parts) by auxiliary fields. We will illustrate here how it works by an explicit calculation. Namely, we compare the reduced Hamiltonians obtained by *(i)* Hamiltonian reduction from the original one and *(ii)* by the Legendre transformation from the reduced Lagrangian and show that they coincide.

Our starting point is the complex sigma model with the coordinates

$$z^m = x^m + iy^m\,, \qquad \bar{z}^{\bar{m}} = x^m - iy^m\,. \tag{A.1}$$

The metric tensor $h_{m\bar{n}}$ is Hermitian, but not necessarily real,

$$h_{m\bar{n}} = \frac{1}{2}\left(g_{(mn)} + ib_{[mn]}\right)\,. \tag{A.2}$$

In the case when the real tensors $g_{(mn)}$ and $b_{[mn]}$ and other structures in the Hamiltonian do not depend on the imaginary parts y^m, we can perform the Hamiltonian reduction. Disregard for simplicity the fermion variables (the recipe $\dot{y}^m \to B^M$ works actually not only for SQM where it is justified by comparing the supertransformation laws before and after reduction, but also for purely bosonic systems) and consider the Lagrangian

$$L = h_{m\bar{n}}\dot{z}^m\dot{\bar{z}}^{\bar{n}} + G_m\dot{z}^m + \bar{G}_{\bar{m}}\dot{\bar{z}}^{\bar{m}} - V\,, \tag{A.3}$$

where G_m, $\bar{G}_{\bar{m}}$ and V do not depend on the imaginary parts y^m. [15]

The corresponding Hamiltonian is

$$H = (\bar{\pi}_{\bar{n}} - \bar{G}_{\bar{n}})(h^{-1})^{\bar{n}m}(\pi_m - G_m) + V\,. \tag{A.4}$$

We represent now

$$\pi_m = \tfrac{1}{2}\left(p_m^{(x)} - ip_m^{(y)}\right), \qquad \bar{\pi}_{\bar{m}} = \tfrac{1}{2}\left(p_m^{(x)} + ip_m^{(y)}\right). \tag{A.5}$$

and perform the reduction. The reduced Hamiltonian is

$$H^{\rm red} \;=\; \tfrac{1}{4}\,(h^{-1})^{NM}(p_N - 2\bar{G}_N)(p_M - 2G_M) + V\,, \tag{A.6}$$

[15] We need not be concerned with their nature, though one can also note that, in the Dolbeault model we are mostly interested in here (HKT models represent their particular case), G_m is associated with the gauge potential. In the full Lagrangian that also includes fermions, G_m contains in addition a bilinear in fermions term.

where $(h^{-1})^{NM} \equiv (h^{-1})^{\bar{n}m}$. We are now using capital Latin indices and are not displaying anymore the superscript $^{(x)}$ for p.

On the other hand, the reduced *Lagrangian* is obtained from (A.3) by substituting $\dot{y}^m \to B^M$ (B^M being the real auxiliary fields) and reads

$$L^{\rm red} \;=\; h_{MN}(\dot{x}^M + iB^M)(\dot{x}^N - iB^N) + G_M(\dot{x}^M + iB^M) + \bar{G}_M(\dot{x}^M - iB^M) - V\,. \tag{A.7}$$

Representing

$$G_M = \tfrac{1}{2}\,(R_M + iM_M)\,, \qquad \bar{G}_M = \tfrac{1}{2}\,(R_M - iM_M)\,. \tag{A.8}$$

and excluding B^M, we obtain

$$L^{\rm red} \;=\; \frac{1}{2}\,G_{MN}\dot{x}^M\dot{x}^N + \left[R_M + b_{MK}(g^{-1})^{KN}M_N\right]\dot{x}^M - \tfrac{1}{2}\,(g^{-1})^{MN}M_M M_N - V\,, \tag{A.9}$$

where

$$G_{MN} = g_{MN} + b_{MK}(g^{-1})^{KL}b_{LN}\,. \tag{A.10}$$

Bearing in mind that the tensor $(h^{-1})^{NM}$ entering (A.6) is expressed as

$$(h^{-1})^{NM} = 2\left[(G^{-1})^{NM} - i(G^{-1})^{NK}b_{KL}(g^{-1})^{LM}\right]\,, \tag{A.11}$$

it is a straightforward exercise to verify that (A.6) and (A.9) are related to each other by the standard Legendre transformation.

Appendix B: Component Lagrangians

B.1. Multiplets $(\mathbf{4},\mathbf{4},\mathbf{0})$

Superfield action (4.19) of interacting linear $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplets yields the component Lagrangian $L = L_b + L_{2f} + L_{4f}$,

$$L_b \;=\; \left(\Delta^{\alpha\beta}_{m\bar{n}}\mathcal{L}\right)\dot{v}^m_\alpha\dot{\bar{v}}^{\bar{n}}_\beta\,, \tag{B.1}$$

$$\begin{aligned} L_{2f} \;=\;& \tfrac{i}{2}\left(\Delta^{\alpha\beta}_{m\bar{n}}\mathcal{L}\right)\left(\psi^m_\alpha\dot{\bar{\psi}}^{\bar{n}}_\beta - \dot{\psi}^m_\alpha\bar{\psi}^{\bar{n}}_\beta\right) \\ &+i\left(\partial^\gamma_k\Delta^{\alpha\beta}_{m\bar{n}}\mathcal{L}\right)\dot{v}^m_\alpha\bar{\psi}^{\bar{n}}_\beta\psi^k_\gamma - i\left(\bar{\partial}^\gamma_{\bar{k}}\Delta^{\alpha\beta}_{m\bar{n}}\mathcal{L}\right)\bar{\psi}^{\bar{k}}_\gamma\psi^m_\alpha\dot{\bar{v}}^{\bar{n}}_\beta \\ &+\tfrac{i}{2}\left(\dot{v}^k_\gamma\partial^\gamma_k - \dot{\bar{v}}^{\bar{k}}_\gamma\bar{\partial}^\gamma_{\bar{k}}\right)\left(\Delta^{\alpha\beta}_{m\bar{n}}\mathcal{L}\right)\psi^m_\alpha\bar{\psi}^{\bar{n}}_\beta\,, \end{aligned} \tag{B.2}$$

$$L_{4f} \;=\; \tfrac{1}{4}\,\epsilon_{mp}\,\epsilon_{\bar{k}\bar{r}}\left(\Delta^{\gamma\delta}_{r\bar{l}}\,\Delta^{\beta\alpha}_{n\bar{p}}\mathcal{L}\right)\psi^m_\alpha\psi^n_\beta\bar{\psi}^{\bar{k}}_\gamma\bar{\psi}^{\bar{l}}_\delta\,, \tag{B.3}$$

where $\partial^\alpha_m = \partial/\partial v^m_\alpha$, $\bar{\partial}^\alpha_{\bar{m}} = \partial/\partial\bar{v}^{\bar{m}}_\alpha$ and $\Delta^{\alpha\beta}_{m\bar{n}}\mathcal{L}$ is defined by (4.20).

B.2. Multiplets $(\mathbf{2},\mathbf{4},\mathbf{2})$

The component Lagrangian $\tilde{L}=\tilde{L}_b+\tilde{L}_{2f}+\tilde{L}_{4f}$ of the superfield action (4.38) of interacting linear chiral $(\mathbf{2},\mathbf{4},\mathbf{2})$ multiplets has the following form

$$\tilde{L}_b \;=\; (\partial_\alpha\bar{\partial}_{\bar\beta}\mathcal{K})\left(\dot{z}^\alpha\dot{\bar{z}}^{\bar\beta}\;+\;A^\alpha\bar{A}^{\bar\beta}\right)-\mathcal{C}_{[\alpha\beta]}\,\dot{z}^\alpha A^\beta-\bar{\mathcal{C}}_{[\bar\alpha\bar\beta]}\,\dot{\bar{z}}^{\bar\alpha}\bar{A}^{\bar\beta}\,, \tag{B.4}$$

$$\begin{aligned}\tilde{L}_{2f} \;=\;& \tfrac{i}{2}\,(\partial_\alpha\bar{\partial}_{\bar\beta}\mathcal{K})\left(\bar{\phi}^{\bar\beta}\dot{\phi}^\alpha-\dot{\bar{\phi}}^{\bar\beta}\phi^\alpha+\bar{\varphi}^{\bar\beta}\dot{\varphi}^\alpha-\dot{\bar{\varphi}}^{\bar\beta}\varphi^\alpha\right)\\ &-\tfrac{1}{2}\,\mathcal{C}_{[\alpha\beta]}\left(\varphi^\alpha\dot{\phi}^\beta-\phi^\alpha\dot{\varphi}^\beta\right)+\tfrac{1}{2}\,\bar{\mathcal{C}}_{[\bar\alpha\bar\beta]}\left(\bar{\varphi}^{\bar\alpha}\dot{\bar{\phi}}^{\bar\beta}-\bar{\phi}^{\bar\alpha}\dot{\bar{\varphi}}^{\bar\beta}\right)\\ &-\tfrac{i}{2}\left[\left(\partial_\alpha\partial_\beta\bar{\partial}_{\bar\gamma}\mathcal{K}\right)\dot{z}^\alpha-\left(\bar{\partial}_{\bar\alpha}\partial_\beta\bar{\partial}_{\bar\gamma}\mathcal{K}\right)\dot{\bar{z}}^{\bar\alpha}\right]\left(\phi^\beta\bar{\phi}^{\bar\gamma}+\varphi^\beta\bar{\varphi}^{\bar\gamma}\right)\\ &-\left(\partial_\alpha\bar{\partial}_{\bar\beta}\bar{\partial}_{\bar\gamma}\mathcal{K}\right)A^\alpha\bar{\phi}^{\bar\beta}\bar{\varphi}^{\bar\gamma}+\left(\bar{\partial}_{\bar\alpha}\partial_\beta\partial_\gamma\mathcal{K}\right)\bar{A}^{\bar\alpha}\phi^\beta\varphi^\gamma\\ &-\tfrac{1}{2}\,\phi^\gamma\left(\partial_\gamma\mathcal{C}_{[\alpha\beta]}\right)\dot{z}^\alpha\varphi^\beta+\tfrac{1}{2}\,\varphi^\gamma\left(\partial_\gamma\mathcal{C}_{[\alpha\beta]}\right)\dot{z}^\alpha\phi^\beta\\ &+\tfrac{1}{2}\,\bar{\phi}^{\bar\gamma}\left(\bar{\partial}_{\bar\gamma}\bar{\mathcal{C}}_{[\bar\alpha\bar\beta]}\right)\dot{\bar{z}}^{\bar\alpha}\bar{\varphi}^{\bar\beta}-\tfrac{1}{2}\,\bar{\varphi}^{\bar\gamma}\left(\bar{\partial}_{\bar\gamma}\bar{\mathcal{C}}_{[\bar\alpha\bar\beta]}\right)\dot{\bar{z}}^{\bar\alpha}\bar{\phi}^{\bar\beta}\,,\end{aligned} \tag{B.5}$$

$$\tilde{L}_{4f}=-\left(\partial_\alpha\partial_\beta\bar{\partial}_{\bar\gamma}\bar{\partial}_{\bar\delta}\mathcal{K}\right)\phi^\alpha\varphi^\beta\bar{\phi}^{\bar\gamma}\bar{\varphi}^{\bar\delta}\,, \tag{B.6}$$

where $\partial_\alpha\equiv\partial/\partial z^\alpha$, $\bar{\partial}_{\bar\alpha}\equiv\partial/\partial\bar{z}^{\bar\alpha}$ and $\mathcal{C}_{\alpha\beta}=\partial_\alpha(z^\gamma\mathcal{F}_{\gamma\beta})$, $\bar{\mathcal{C}}_{\bar\alpha\bar\beta}=\partial_{\bar\alpha}(\bar{z}^{\bar\gamma}\bar{\mathcal{F}}_{\bar\gamma\bar\beta})$. Note the identities

$$\partial_\alpha\mathcal{C}_{[\beta\gamma]}+\partial_\beta\mathcal{C}_{[\gamma\alpha]}+\partial_\gamma\mathcal{C}_{[\alpha\beta]}=0\,,\qquad \bar{\partial}_{\bar\alpha}\bar{\mathcal{C}}_{[\bar\beta\bar\gamma]}+\bar{\partial}_{\bar\beta}\bar{\mathcal{C}}_{[\bar\gamma\bar\alpha]}+\bar{\partial}_{\bar\gamma}\bar{\mathcal{C}}_{[\bar\alpha\bar\beta]}=0\,. \tag{B.7}$$

It is instructive to compare the Lagrangian (B.4) - (B.6) with the Lagrangian of a *generic* quasicomplex $\mathcal{N}=2$ model derived in [1]. The expression of the Lagrangian (4.9) into the components of (4.6) reads

$$\begin{aligned}L=&\frac{1}{2}\,g_{(MN)}\big(\dot{x}^M\dot{x}^N+B^MB^N\big)+b_{[MN]}\dot{x}^MB^N+\frac{i}{2}\,g_{(MN)}\big(\bar{\chi}^N\nabla\chi^M-\nabla\bar{\chi}^N\chi^M\big)\\ &-\frac{1}{2}\,b_{[MN]}\big(\bar{\chi}^N\dot{\chi}^M-\dot{\bar{\chi}}^N\chi^M\big)-\frac{1}{2}\,\partial_P\partial_Q\big(g_{(MN)}+ib_{[MN]}\big)\chi^M\bar{\chi}^N\chi^P\bar{\chi}^Q\\ &+G_{M,PQ}B^M\chi^P\bar{\chi}^Q-\frac{1}{2}\left(\partial_Mb_{[NP]}+\partial_Nb_{[MP]}\right)\dot{x}^P\chi^M\bar{\chi}^N\,,\end{aligned} \tag{B.8}$$

$$G_{M,PQ}=\Gamma_{M,PQ}-\frac{i}{2}\left(\partial_Mb_{[PQ]}+\partial_Pb_{[QM]}+\partial_Qb_{[MP]}\right)\,, \tag{B.9}$$

with $\Gamma_{M,PQ}$ being the standard Christoffels for $g_{(MN)}$,

$$\Gamma_{M,PQ}=\frac{1}{2}\big[\partial_Pg_{(MQ)}+\partial_Qg_{(MP)}-\partial_Mg_{(PQ)}\big]\,, \tag{B.10}$$

and

$$\nabla\psi^M=\dot{\psi}^M+\Gamma^M_{NQ}\dot{x}^N\psi^Q\,. \tag{B.11}$$

One can observe that the Lagrangian (B.8) involves among other terms the b-dependent 4-fermion term $\sim b\chi\bar{\chi}\chi\bar{\chi}$ and the terms $(\partial b)F\chi\bar{\chi}$, which do not have a counterpart in (B.5), (B.6). Well, one can explicitly show that, in the $\mathcal{N}=4$ case for *particular* $b_{[MN]}$ depending on holomorphic $C_{\alpha\beta}$, these contributions vanish, indeed.

Appendix C: Reduction of general HKT models

We will construct here a generic form of the HKT prepotential in (4.13) allowing reduction and show that the bosonic action of the reduced model coincides with (4.30) with generic $\mathcal{F}_{\alpha\beta}$.

We define the superfields

$$\mathcal{V}^m_\alpha \;=\; \mathcal{X}^m_\alpha + i\mathcal{Y}^m_\alpha, \qquad \mathcal{Z}^\alpha \;=\; \mathcal{X}^1_\alpha + i\mathcal{X}^2_\alpha, \qquad \Xi^\alpha \;=\; \mathcal{Y}^1_\alpha + i\mathcal{Y}^2_\alpha \tag{C.1}$$

with bosonic component fields

$$v^m_\alpha \;=\; x^m_\alpha + iy^m_\alpha, \qquad z^\alpha \;=\; x^1_\alpha + ix^2_\alpha, \qquad \xi^\alpha \;=\; y^1_\alpha + iy^2_\alpha\,. \tag{C.2}$$

We consider now the operator $\Delta^{\alpha\beta}_{m\bar n}$ entering (4.20) and express it in terms of $z, \bar z, \xi, \bar\xi$,

$$\begin{aligned}
2\Delta^{\alpha\beta}_{1\bar 1} \;&=\; \frac{\partial^2}{\partial z^\alpha \partial \bar z^\beta} + \frac{\partial^2}{\partial z^\beta \partial \bar z^\alpha} + i\left(\frac{\partial^2}{\partial z^\alpha \partial \xi^\beta} - \frac{\partial^2}{\partial z^\beta \partial \xi^\alpha} + \frac{\partial^2}{\partial \bar z^\alpha \partial \bar\xi^\beta} - \frac{\partial^2}{\partial \bar z^\beta \partial \bar\xi^\alpha}\right) \\
&\qquad + \frac{\partial^2}{\partial \xi^\alpha \partial \bar\xi^\beta} + \frac{\partial^2}{\partial \xi^\beta \partial \bar\xi^\alpha} \\
2\Delta^{\alpha\beta}_{2\bar 2} \;&=\; \frac{\partial^2}{\partial z^\alpha \partial \bar z^\beta} + \frac{\partial^2}{\partial z^\beta \partial \bar z^\alpha} - i\left(\frac{\partial^2}{\partial z^\alpha \partial \xi^\beta} - \frac{\partial^2}{\partial z^\beta \partial \xi^\alpha} + \frac{\partial^2}{\partial \bar z^\alpha \partial \bar\xi^\beta} - \frac{\partial^2}{\partial \bar z^\beta \partial \bar\xi^\alpha}\right) \\
&\qquad + \frac{\partial^2}{\partial \xi^\alpha \partial \bar\xi^\beta} + \frac{\partial^2}{\partial \xi^\beta \partial \bar\xi^\alpha} \\
2\Delta^{\alpha\beta}_{1\bar 2} \;&=\; i\left(\frac{\partial^2}{\partial \bar z^\alpha \partial z^\beta} - \frac{\partial^2}{\partial z^\alpha \partial \bar z^\beta}\right) + \frac{\partial^2}{\partial \xi^\alpha \partial z^\beta} - \frac{\partial^2}{\partial z^\alpha \partial \xi^\beta} + \frac{\partial^2}{\partial \bar z^\alpha \partial \bar\xi^\beta} - \frac{\partial^2}{\partial \bar z^\beta \partial \bar\xi^\alpha} \\
&\qquad + i\left(\frac{\partial^2}{\partial \bar\xi^\alpha \partial \xi^\beta} - \frac{\partial^2}{\partial \xi^\alpha \partial \bar\xi^\beta}\right) \\
2\Delta^{\alpha\beta}_{2\bar 1} \;&=\; i\left(\frac{\partial^2}{\partial z^\alpha \partial \bar z^\beta} - \frac{\partial^2}{\partial \bar z^\alpha \partial z^\beta}\right) + \frac{\partial^2}{\partial \xi^\alpha \partial z^\beta} - \frac{\partial^2}{\partial z^\alpha \partial \xi^\beta} + \frac{\partial^2}{\partial \bar z^\alpha \partial \bar\xi^\beta} - \frac{\partial^2}{\partial \bar z^\beta \partial \bar\xi^\alpha} \\
&\qquad - i\left(\frac{\partial^2}{\partial \bar\xi^\alpha \partial \xi^\beta} - \frac{\partial^2}{\partial \xi^\alpha \partial \bar\xi^\beta}\right).
\end{aligned} \tag{C.3}$$

The second term in (4.26) is expressed via $\mathcal{Z}, \bar{\mathcal{Z}}, \Xi, \bar\Xi$ as

$$-\,\tfrac{1}{2}\,\mathcal{C}_{\alpha\beta}\left(\mathcal{Z}^\alpha \Xi^\beta + \bar{\mathcal{Z}}^\alpha \bar\Xi^\beta\right). \tag{C.4}$$

It is linear in $\xi, \bar\xi$, but the result of the action of (C.3) on (C.4) gives a constant not depending on the imaginary parts of v^m_α entering $\xi, \bar\xi$.

We generalize now (C.4) by introducing the following term in the prepotential

$$\Delta\mathcal{L} = -\tfrac{1}{2}\left[\mathcal{F}_{\alpha\beta}(\mathcal{Z})\,\mathcal{Z}^\alpha \Xi^\beta + \bar{\mathcal{F}}_{\alpha\beta}(\bar{\mathcal{Z}})\,\bar{\mathcal{Z}}^\alpha \bar\Xi^\beta\right]. \tag{C.5}$$

It is not difficult to observe that only the mixed terms in (C.3) involving both z and ξ derivatives give a nonzero result when acting on (C.5). The result does not depend on $\xi, \bar\xi$ and is expressed in the form (4.30). *Q.E.D.*

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